Toric reflection groups

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The symmetric group $\mathfrak{S}_3$ and various constructions of its braid group B_3

- The symmetric group $\mathfrak{S}_3$ is a quotient of the 3-strand braid group B_3:

$$\mathfrak{S}_3 = \langle\ s_1, s_2 \mid s_1^2 = 1 = s_2^2,\ s_1 s_2 s_1 = s_2 s_1 s_2\ \rangle$$
$$B_3 = \langle\ \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\ \rangle$$

- The group B_3 is obtained from $\mathfrak{S}_3$ by removing the relations $s_1^2 = 1 = s_2^2$ in the above presentation.
  "Artin-Tits group"
- The group $\mathfrak{S}_3$ also acts by reflections on a 2-dim. (real, or complex) vector space V, and $B_3 = \pi_1(V_{\text{reg}}/W)$.
  "Complex braid group"
- The group B_3 is also the fundamental group of the complement of the trefoil knot in $\mathbb{R}^3$. Does this fit into a more general theory as for the above two situations ?

Torus knot groups and reflection groups

- Let $n, m \geq 2$, $n < m$, $(n,m) = 1$. The *torus knot group* $G(n,m)$ can be defined by the presentation

$$\langle\, x_1, x_2, \ldots, x_n \mid \underbrace{x_1 x_2 \cdots}_{m \text{ factors}} = \underbrace{x_2 x_3 \cdots}_{m \text{ factors}} = \cdots = \underbrace{x_n x_1 \cdots}_{m \text{ factors}} \,\rangle$$

- Example: for $n = 2$ and $m = 3$ we have $G(2,3) \cong B_3$. The finite CRG with braid group a torus knot group:

W	B_W
$\mathfrak{S}_3, G_4, G_8, G_{16}$	$G(2,3) = B_3$
$I_2(\ell)$ with odd ℓ	$G(2,\ell) =$ Artin group of type $I_2(\ell)$
G_{12}	$G(3,4)$
G_{20}	$G(2,5) =$ Artin group of type $I_2(5)$
G_{22}	$G(3,5)$

- W is obtained from $B_W = G(n,m)$ by adding rel. of the form $x_i^k = 1$ for some $k \geq 2$. We get all the finite CRG of rank 2 with a single conjugacy class of reflecting hyperplanes.

Toric reflection groups

- Question: Let $n, m \geq 2$ as above, $k \geq 2$. What can be said about the group $W(k, n, m)$ with presentation

$$\left\langle x_1, x_2, \dots, x_n \;\middle|\; \begin{array}{c} x_i^k = 1 \text{ for } i = 1, \dots, n, \\ \underbrace{x_1 x_2 \cdots}_{m \text{ factors}} = \underbrace{x_2 x_3 \cdots}_{m \text{ factors}} = \cdots = \underbrace{x_n x_1 \cdots}_{m \text{ factors}} \end{array} \right\rangle.$$

- More precise questions:
  - When is $W(k, n, m)$ finite ?
  - Does $W(k, n, m)$ admit a structure of (complex) reflection group (of rank two) in any reasonable sense? If yes, can we classify them as reflection groups?
  - Is $G(n, m)$ the "braid group" of $W(k, n, m)$?
- Example (Coxeter 1957): the group $W(k, 2, 3)$, i.e., the quotient of the three-strand braid group B_3 by the relations $\sigma_1^k = 1 = \sigma_2^k$, is finite if and only if $k \leq 5$.
- We call a group of the form $W(k, n, m)$ a *toric reflection group*.

NOTE

The invited lecture by A. D. Alexandrov, "Uniqueness Theorem for Surfaces in the Large," has been enlarged by the author and is to be published elsewhere. It is therefore not included in these Proceedings.

FACTOR GROUPS OF THE BRAID GROUP[1]

H. S. M. COXETER, *University of Toronto*

Introduction. The relation $R_1R_2 = R_2R_1$, or

$$R_1 \rightleftarrows R_2,$$

which says that two elements commute, has been studied ever since 1852, when Hamilton first recognized the possibility of denying it. If R_1 and R_2 commute, R_2 transforms R_1 into itself; thus a natural generalization is the relation

$$R_1R_2R_1 = R_2R_1R_2,$$

which says that R_2R_1 transforms R_1 into R_2. In 1926, Artin considered a sequence of elements $R_1, R_2, \ldots, R_{n-1}$, in which consecutive members are so related while non-consecutive members commute. He observed that such elements of period 2 generate the symmetric group $\mathfrak{S}_n$. The chief purpose of this paper is to consider the effect of changing the period of the generators from 2 to p. Representing the generators by unitary reflections, we find (in § 12) that the order is changed from $n!$ to

$$(\tfrac{1}{2}V)^{n-1}\, n!,$$

where V is the number of vertices of the regular polyhedron or tessellation $\{p, n\}$.

As a by-product we obtain, for the simple group of order 25920, the presentation 5.5 or

$$R^5 = R_1^3 = (RR_1)^4 = E, \qquad R_1 \rightleftarrows R^{-2}R_1R^2,$$

which is more concise than that of Dickson (**15**, pp. 293, 296).

1. Artin's braid group. The simplest braid, say E, consists of n vertical strands (or strings) joining two horizontal rows of n points (or pegs). Other n-strand braids are variants of this: the strands remain vertical in general, but at certain levels two neigh-

[1]Two lectures (§1–6 and 7–12) delivered at Banff, September 5 and 6, 1957.

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19. F. Klein, *Lectures on the icosahedron* (London, 1913).
20. G. A. Miller, *Collected works*, vol. 2 (Urbana, 1938).
21. ———, *Collected works*, vol. 3 (Urbana, 1946).
22. A. F. Möbius, *Gesammelte Werke*, vol. 1 (Leipzig, 1886).
23. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.
24. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math., *6* (1954), 274–304.
25. O. Veblen and J. W. Young, *Projective geometry*, vol. 1 (Boston, 1910).

QUELQUES PROBLÈMES ACTUELS CONCERNANT L'ENSEIGNEMENT MATHÉMATIQUE EN FRANCE

J. DIXMIER, *University of Paris*

Si vous aimez le changement, je vous conseille d'aller en France et d'y devenir professeur de mathématiques. Tous les trois mois, l'organisation de l'enseignement est modifiée. Depuis deux ans, un nouvel examen d'entrée dans les facultés des sciences françaises a été créé; les méthodes de travail dans les classes primaires et l'examen d'entrée dans les lycées ont été changés; un nouveau cycle d'enseignement, dit "de recherche" est apparu dans les facultés des sciences; de nouveaux programmes sont appliqués dans les classes secondaires de mathématiques spéciales; l'enseignement technique se développe considérablement; l'an prochain, les horaires des classes de mathématiques dans les lycées vont changer; un nouveau système de recrutement des professeurs sera mis en place; les programmes de licence seront réformés. Dominant tout cela, la réforme générale de l'enseignement, qui fournit depuis dix ans et plus des sujets de controverse, semble approcher de sa réalisation.

Avant d'examiner cette situation en détail, je crois qu'il est utile de bien nous entendre sur le sens de certains mots qui appartiennent au vocabulaire scolaire français. Comme vous le savez sans doute, l'enseignement comporte chez nous trois étages superposés: l'enseignement primaire, le secondaire et le supérieur. On appelle école (tout simplement), l'établissement où les enfants reçoivent l'enseignement primaire; on appelle lycée (ou dans certains cas collège), l'établissement propre à l'enseignement secondaire, et on appelle faculté celui que fréquentent les étudiants de l'enseignement supérieur. Dans les facultés des lettres, des sciences et de droit, on délivre aux étudiants plusieurs sortes de diplômes, dont les plus importants sont la licence et le doctorat. Le mot université s'applique chez nous à l'ensemble administratif de tous les établissements scolaires publics.

Je disais tout à l'heure que l'enseignement des mathématiques

J-groups

Achar and Aubert introduced a family of (in general infinite) groups, called *J-groups*.

Let a, b, c three integers ≥ 1. Let $J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}$ be the group defined by the presentation

$$\langle\ s, t, u \mid s^a = t^b = u^c = 1,\ stu = tus = ust\ \rangle$$

Let a', b' and c' be three pairwise coprime integers, dividing a, b and c respectively. Let $J\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ be the normal subgroup of $J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}$ generated by $s^{a'}, t^{b'}$ and $u^{c'}$. We omit 1's in the second row of parameters, consistently with

$$J\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \end{pmatrix} = J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}.$$

Toric reflection groups are J-groups

Theorem (Achar-Aubert, 2008)

A J-group is finite if and only if it is a finite complex reflection group of rank 2.

- Achar and Aubert also showed that every J-group G admits a rep. $\rho : G \longrightarrow \mathrm{GL}_2(\mathbb{C})$, where $\rho(s)$, $\rho(t)$ and $\rho(u)$ are reflections preserving a Hermitian form (so that the image is a CRG). When G is not finite, this representation is *not* faithful in general.
- This result is somewhat reminiscent of the following theorem : let W be a Coxeter group. Then W is a real reflection group if and only if W is finite.

Theorem (Toric reflection groups are J-groups)

Let k, n, m be three integers ≥ 2 with $n < m$ and n, m coprime. We have $W(k,n,m) \cong J\begin{pmatrix} k & n & m \\ & n & m \end{pmatrix}$.

Reflection group structure on $W(k, n, m)$

Corollary

A toric reflection group is finite if and only if it is a finite CRG of rank two with a single conjugacy class of reflecting hyperplanes (we listed all of them above).

- The isom. above maps conjugates of nontrivial powers of s, t and u to conjugates of nontrivial powers of the x_i's. In the case where $W(k, n, m)$ is finite, these are precisely the reflections in $W(k, n, m)$.
- Call an element of $W(k, n, m)$ a *reflection* if it is a conjugate of a nontrivial power of some x_i.
- In this way we can put a "reflection-like" group structure on $W(k, n, m)$.

Center of toric reflection groups and classification

- It is known that the center of $G(n, m)$ is infinite cyclic, generated by $(x_1 x_2 \cdots x_n)^m$ (Schreier, 1923).
- Let $c = (x_1 x_2 \cdots x_n)^m$ be the image of the above element in $W(k, n, m)$.
- Denote by $W_{k,n,m}$ the rank-three Coxeter group

$$\left\langle r_1, r_2, r_3 \;\middle|\; \begin{array}{c} r_1^2 = r_2^2 = r_3^2 = 1, \\ (r_1 r_2)^k = (r_2 r_3)^n = (r_3 r_1)^m = 1 \end{array} \right\rangle$$

- Denote by $W_{k,n,m}^+$ the alternating subgroup of $W_{k,n,m}$, i.e., the kernel of the homomorphism $W_{k,n,m} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, $r_i \mapsto 1$.

Center of toric reflection groups and classification, II

Theorem

1. *There is a short exact sequence*

$$1 \longrightarrow \langle c \rangle \longrightarrow W(k,n,m) \longrightarrow W^+_{k,n,m} \longrightarrow 1.$$

2. *We have* $Z(W^+_{k,n,m}) = \{1\}$.

Corollary

The center of $W(k,n,m)$ *is cyclic, generated by* c.

Questions

1. *When* $W(k,n,m)$ *is infinite, is* c *of finite order?*
2. *Does* $W(k,n,m)$ *have a solvable word problem?*

Center of toric reflection groups and classification, III

- Say that two toric reflection groups $W(k,n,m)$ and $W(k',n',m')$ are *isomorphic as reflection groups* ($\cong_{\mathrm{ref}}$) if there is an isomorphism φ between them such that both φ and φ^{-1} map reflections to reflections.

Theorem

Let $W(k,n,m)$ and $W(k',n',m')$ be two toric reflection groups. Then $W(k,n,m) \cong_{\mathrm{ref}} W(k',n',m')$ if and only if $(k,n,m) = (k',n',m')$.

Corollary

Let $W = W(k,n,m)$. Define B_W as $G(n,m)$. Then B_W is well-defined, i.e., only depends on the reflection group structure of $W(k,n,m)$. Moreover, if W is finite, then B_W is the complex braid group of $W(k,n,m)$.

Example and questions

- Consider $W(6,2,3)$, i.e., Coxeter's truncated braid group at $k=6$ (it is an infinite group)

$$\langle\ s,t \mid s^6 = 1 = t^6,\ sts = tst\ \rangle$$

- The map to

$$W_{6,2,3} = \left\langle r_1, r_2, r_3 \,\middle|\, \begin{array}{c} r_1^2 = r_2^2 = r_3^2 = 1, \\ (r_1r_2)^6 = (r_2r_3)^3 = (r_3r_1)^2 = 1 \end{array} \right\rangle$$

  sends s to r_1r_2 and t to $r_2r_1r_2r_3$.
- We do not know if $c = (sts)^2 = (st)^3$ has finite order or not in $W(6,2,3)$ (equiv., if st has finite order or not).
- The group $W(6,2,3)$ has no faithful two-dimensional complex reflection representation. Hence one cannot define B_W as $\pi_1(V_{\text{reg}}/W)$.
- Questions: Is there a geometric definition of the braid group of a TRG ? Does c have finite order ? Do TRG have a solvable word problem ?

Thank you for your attention!