On some lattices arising in combinatorial group theory

Thomas Gobet

Institut Denis Poisson, Université de Tours,

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Presented monoids

- Let S be a finite alphabet $\{a, b, ...\}$. Let S^* be the free monoid on S, that is, the monoid of words of finite length in S, where product $=$ concatenation.
- Let $\{u_i\}_{i=1}^k$, $\{w_i\}_{i=1}^k$ be two lists of elements of S^*.
- Consider the set-theoretic quotient M of S^* by the equivalence relation $aw_ib \sim au_ib$ whenever $a, b \in S^*$, and $i \in \{1, 2, \ldots, k\}$.
- This relation is compatible with the concatenation of words in S^*, thus M is still a monoid. It is often denoted $M = \langle S \mid R \rangle$, where S is its generating set S, and R denotes the set $\{w_i = u_i\}_{i=1}^k$ of defining relations. The data $\langle S \mid R \rangle$ is a *presentation* of M.

Example

Let $M_1 = \langle a, b \mid aba = bab \rangle$. In M we have

$$b^2ab = b(bab) = b(aba) = (bab)a = (aba)a = aba^2.$$

Word problem in a finitely presented monoid

- A monoid M is said to be *finitely presented* if there are S, R as in the previous slide such that $M \cong \langle S \mid R \rangle$.

Definition

A finitely presented monoid M is said to have *solvable word problem* if there is an algorithm allowing one to determine in finite time if any two words $x_1, x_2 \in S^*$ represent the same element of M or not.

Example

Let $M_1 = \langle a, b \mid aba = bab \rangle$. There is a unique defining relation $aba = bab$, which preserves the length of words. The word problem is thus trivial: given a word $x_1 \in S^*$, look at all possible ways to apply a defining relation to x_1. It gives a (possible empty) new finite set of words $\{y_1, y_2, \ldots, y_\ell\}$ of the same length. Iterate, until you get no new words. Since the set of words of a given length is finite, it terminates.

Word problem in a finitely presented monoid, II

- If M is equipped with a length function $\lambda : M \longrightarrow \mathbb{Z}_{\geq 0}$ such that $\lambda(ab) = \lambda(a) + \lambda(b)$ and $a \neq 1 \Rightarrow \lambda(a) \neq 0$, then M has a solvable word problem.
- In such a monoid, there is no nontrivial invertible element $\neq 1$. Moreover, the left-divisibility relation $\leq_L$ defines a partial order on M.
- We will only consider such monoids in this talk (Monoids with "Noetherian divisibility").

Example

Consider $M_4 = \langle x, y \mid xyx = y^2 \rangle$. Then M_4 has Noetherian divisibility, with $\lambda(x) = 1$, $\lambda(y) = 2$.

Word problem in finitely presented groups

- The same question can be asked for a finitely presented group $G = \langle S \mid R \rangle$. It is defined as a quotient of a free group $F(S)$ on a finite alphabet S by a finite set of relations R, where words lie in $(S \cup S^{-1})^*$.
- Determining if two words x_1 and x_2 represent the same element of G or not is equivalent to determining if $x_1 x_2^{-1}$ represents 1 or not. Hence an algorithm to determine if a word represents the identity or not is enough.
- There are groups with unsolvable word problem (Novikov, 1955).

Example

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Let $G_1 = \langle a, b \mid aba = bab \rangle$. Since we are in a group, we are allowed to add aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$ at any place of a word without changing the corresponding element, or deleting them when they appear. Hence the word problem becomes much harder...

- Claim: $ab^2a^{-1} = b^{-1}a^2b$.
- Proof:

$$\begin{aligned} ab^2a^{-1} &= (b^{-1}b)ab^2a^{-1} = b^{-1}(bab^2)a^{-1} \\ &= b^{-1}(a^2ba)a^{-1} = b^{-1}a^2b. \end{aligned}$$

Difficulties

- To relate the two words ab^2a^{-1} and $b^{-1}a^2b$ using the defining relations of G_1, we needed to increase the length of the words. There are infinitely many words representing the same element, thus the naïve method which we applied in M_1 cannot be applied in G_1.
- In fact, there are many known solutions to the word problem in G_1, but none of them is trivial. Let us explain the philosophy of one of them. Roughly speaking this will be based on
  - Increasing the number of generators,
  - Reducing the solution of the word problem in G_1 to M_1, where it is trivial.

A particular element of M_1 and G_1

- Consider the element $\Delta = aba = bab \in M_1$. One observes that its set of left and right divisors coincide, and are given by the set $A = \{1, a, b, ab, ba, aba = bab\}$. Hence given $x, y \in A$, there are $u, v \in A$ such that $xu = yv (= \Delta)$. We thus have
$$y^{-1}x = vu^{-1}.$$
- The above property implies that every word in G_1 can be written as a fraction $w_1 w_2^{-1}$, where w_i are positive words in a and b.

Example

Consider the word $b^{-1}ab^{-1}a$. We have $a(ba) = b(ab)$, hence $b^{-1}a = (ab)(ba)^{-1}$. We thus have $b^{-1}ab^{-1}a = (ab)(ba)^{-1}(ab)(ba)^{-1}$. Now we have $(ba)b = (ab)a$, hence $(ba)^{-1}(ab) = ba^{-1}$, yielding

$$b^{-1}ab^{-1}a = abba^{-1}(ba)^{-1} = ab^2a^{-2}b^{-1}.$$

Reducing the word problem in G_1 to the word problem in M_1

- Thus, deciding whether $b^{-1}ab^{-1}a = 1$ is equivalent to deciding whether $ab^2 = ba^2$ in G_1. Note that, this is a priori **not** equivalent to verifying whether $ab^2 = ba^2$ in M_1 or not. But we have:

Theorem (Particular case of a Thm of Garside, 1969)

The natural map $M_1 \longrightarrow G_1$, $a \mapsto a$, $b \mapsto b$ is injective.

- With this theorem, checking whether $ab^2 = ba^2$ becomes trivial. In M_1 (and thus in G_1) we have $ab^2 \neq ba^2$, thus $b^{-1}ab^{-1}a \neq 1$.
- We thus have our algorithm to solve the word problem in G_1: given a word $x_1^{\pm 1} \cdots x_k^{\pm 1}$, where $x_i \in \mathrm{Div}(\Delta)$,
  - Step 1: transform it into a word of the form $y_1^{-1} \cdots y_\ell^{-1} y_{\ell+1} \cdots y_k$.
  - Step 2: check whether $y_\ell y_{\ell-1} \cdots y_1 = y_{\ell+1} \cdots y_k$ in M_1 or not, where the word problem is trivial.

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- A particular element $\Delta \in M$ such that
  1. its set $\mathrm{Div}_L(\Delta)$ of left-divisors coincides with its set $\mathrm{Div}_R(\Delta)$ of right-divisors, thus simply denoted $\mathrm{Div}(\Delta)$,
  2. $|\mathrm{Div}(\Delta)| < \infty$,
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Bad news: In practice, the obtained algorithm is very bad, and it does not give a normal form for the elements of G.

Further assumptions on M

- Since M embeds into G, it is cancellative ($ab = ac \Rightarrow b = c$). If in addition we assume that
  - The poset $(M, \leq_L)$ is a lattice, where $\leq_L$ is the left-divisibility relation,

  Then every fraction $x^{-1}y$ can be reduced into a unique irreducible one $x'^{-1}y'$, by left-killing $\mathrm{gcd}(x, y)$. This yields a normal form, but still hard to calculate in practice, in fact:
- Under the above assumptions, other normal forms can be defined, which are much quicker to calculate in practice (the Garside normal forms).

Garside monoids

Definition (Dehornoy-Paris, 1996)

A *Garside monoid* is a finitely presented monoid $M = \langle S \mid R \rangle$ together with an element $\Delta \in M$, such that

1. M is both left- and right-cancellative,
2. M has Noetherian divisibility,
3. $(M, \leq_L)$ and $(M, \leq_R)$ are lattices,
4. The left- and right-divisors of Δ coincide + form a finite set.
5. The set $\mathrm{Div}(\Delta)$ of divisors of Δ generates M.

- (1) and (3) ensure that $M \hookrightarrow G$, where $G = \langle S \mid R \rangle$.
- (2) ensures that the word problem in M is solvable.
- With (4) and (5), one can define normal forms for elements of G, and they can be calculated using an algorithm which reduces to calculating a sequence of meets and joins in $(\mathrm{Div}(\Delta), \leq_L)$. Hence the WP is solvable in G.

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- $M_3 = \langle x, y \mid x^2 = y^3 \rangle$: set $\Delta = x^2$. Then $\mathrm{Div}_L(\Delta) = \{1, x, y, y^2, \Delta\} = \mathrm{Div}_R(\Delta)$ and restricting the left-divisibility yields a lattice.

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- $M_4 = \langle x, y \mid xyx = y^2 \rangle$: set $\Delta = y^3$. Then $\mathrm{Div}_L(\Delta) = \{1, x, y, y^2, xy, yx, yxy, y^3\} = \mathrm{Div}_R(\Delta)$, and restricting the left-divisibility yields a lattice.

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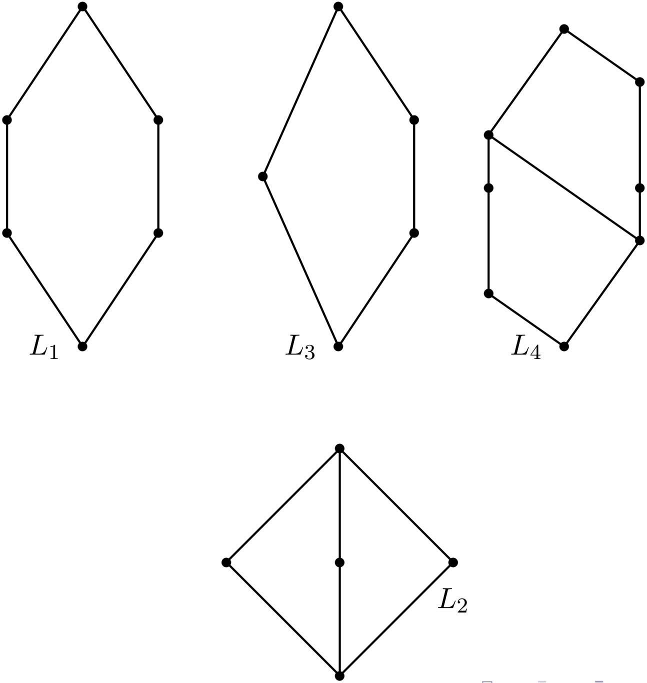
Exercise

Show that $G_1 \cong G_2 \cong G_3 \cong G_4$.

This thus yields four different solutions to the word problem in G_1...

Questions

- (Algebraist) Given a group G, can we classify the monoids M yielding a solution to the word problem as explained above ? (classification of Garside structures on a given group. Completely open even for G_1...)
- (Computational group theorist) Among the solutions which the algebraist above classified, which one provides the best algorithm to solve the word problem in a given group G admitting such structures ?
- (Combinatorist) Can I realise my favorite lattice as the lattice of divisors of a Garside element in a Garside monoid ?
- ...


L_1
L_3
L_4
L_2

Generalization of G_1

- The lattice L_1 is the lattice of permutations in $\mathfrak{S}_3$ ordered by the weak Bruhat order.
- The lattice L_2 is the lattice of (noncrossing) partitions of $\{1, 2, 3\}$.
- In fact, the group G_1 is the 3-stranded braid group B_3. The n-stranded braid group B_n admits the (Garside) presentation with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ \forall i = 1, \dots, n-2,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ whenever } |i-j| > 1,$$

generalizing the presentation of G_1. The Garside element is the positive lift Δ of the longest permutation of $\mathfrak{S}_n$, and the lattice $(\mathrm{Div}(\Delta), \leq_L)$ is isomorphic to the weak Bruhat order on $\mathfrak{S}_n$.

Generalizations of G_2, G_3

- The n-strand braid group B_n is also isomorphic to the group with the (Garside) presentation with generators a_{ij}, $1 \leq i < j \leq n$ and relations

$$a_{ij}a_{jk} = a_{jk}a_{ik} = a_{ik}a_{ij}, \forall 1 \leq i < j < k \leq n,$$
$$a_{ij}a_{kl} = a_{kl}a_{ij}, \forall 1 \leq i < j < k < l \leq n$$
$$\text{or } 1 \leq i < k < l < j \leq n,$$

  generalizing the presentation of G_2. The Garside element is $\Delta = a_{1,2}a_{2,3} \cdots a_{n-1,n}$, and the lattice $(\mathrm{Div}(\Delta), \leq_L)$ is isomorphic to the noncrossing partition lattice NC(n) (Birman-Ko-Lee, 1998).
- The presentation of G_3 generalizes to a family of groups $G(n,m) = \langle x, y \mid x^n = y^m \rangle$ for all $n, m \geq 2$, which yields a Garside presentation. When n and m are coprime $G(n,m)$ is a torus knot group. The lattice is of "spindle" type.

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Theorem (G. 2021)

Let $n \geq 1$. The monoid $M(n)$ with generators $\rho_1, \rho_2, \ldots, \rho_n$ and relations

$$\rho_1 \rho_n \rho_i = \rho_{i+1} \rho_n, \forall i = 1, \ldots, n-1$$

is a Garside presentation. Note that $M(2) = M_4$. The corresponding group is isomorphic to $G(n, n+1)$, which is an extension of B_{n+1} (with isomorphism for $n = 1, 2$). The Garside element is $\Delta_n = \rho_n^{n+1}$.

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- It does not yield an explicit description of the lattice of divisors of Δ_n, and not even a formula for $|\mathrm{Div}(\Delta_n)|$...

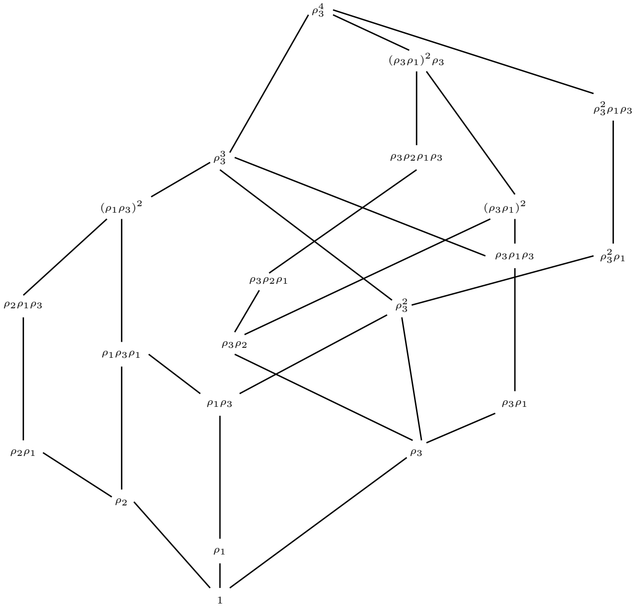


Figure: Lattice of divisors of Δ_3.

First step: number of words for the Garside element

- Before understanding how many divisors $\Delta_n = \rho_n^{n+1}$ has, we need to understand how many words in the alphabet $\{\rho_1, \rho_2, \ldots, \rho_n\}$ represent Δ_n.
- A *Schröder tree* is a rooted plane tree in which every inner vertex has at least two children.
- Consider a Schröder tree T on $n+1$ leaves. We assign to each vertex v of T (except the root) a label $\lambda(v) \in \{1, 2, \ldots, n\}$ as follows:
  - The vertices are labelled in post-order.
  - If v is a leftmost child of a vertex w of T, then w is the root of a Schröder tree $(w, (T_1, \ldots, T_k))$ and v is the root of T_1. Then $\lambda(v)$ is defined to be the number of leaves in the forest $T_2, \ldots, T_k$.
  - If v is not the leftmost child of a vertex of T, we consider $LD(v)$ the set of its leftmost descendants consisting of the leftmost child of v and its leftmost child, etc. Then the label of v is $n - \sum_{w \in LD(v)} \lambda(w)$.

Example

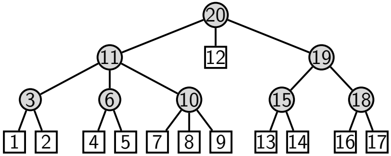


Figure: Post-order on the vertices of a Schröder tree with $11+1$ leaves.

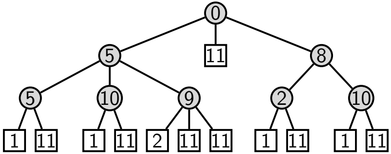


Figure: Labeling of the above Schroeder tree.

Schroeder trees and words for Δ_n

- Define a map Φ from the set $\mathcal{T}(n+1)$ of Schroeder trees on $n+1$ leaves to words in $\{\rho_1, \rho_2, \ldots, \rho_n\}$, which to a Schroeder tree T assigns the word $\rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}$, where $i_1 i_2 \cdots i_k$ is the sequence of labels of T, ordered following the post-order convention.

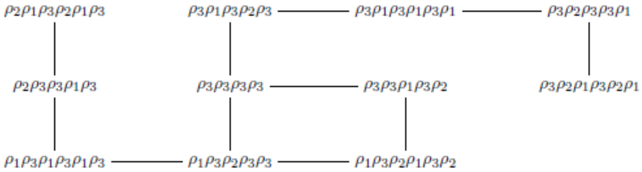
Theorem (Rognerud-G., 2023)

1. *The map Φ has image in the set $W(\Delta_n)$ of words for ρ_n^{n+1} in $M(n)$,*
2. *The map $\Phi : \mathcal{T}(n+1) \longrightarrow W(\Delta_n)$ is bijective.*

Corollary

We have $|W(\Delta_n)| = |\mathcal{T}(n+1)|$, which is equal to the little Schroeder number $S(n+1)$: $S(1) = S(2) = 1$,

$$S(n) = \frac{3(2n-3)S(n-1) - (n-3)S(n-2)}{n}.$$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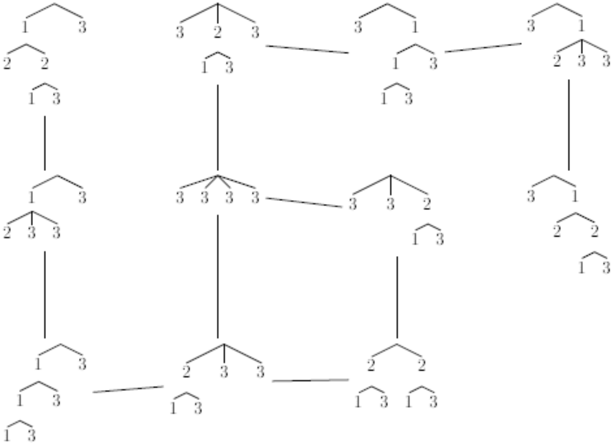

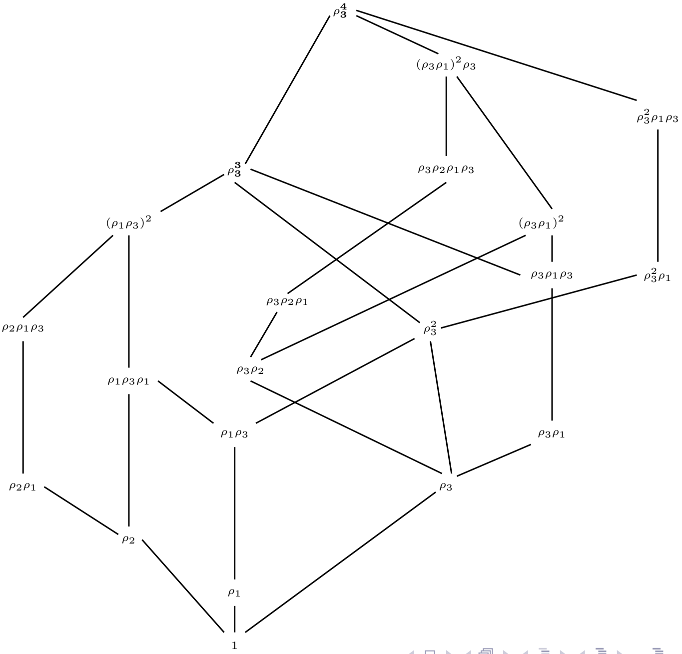
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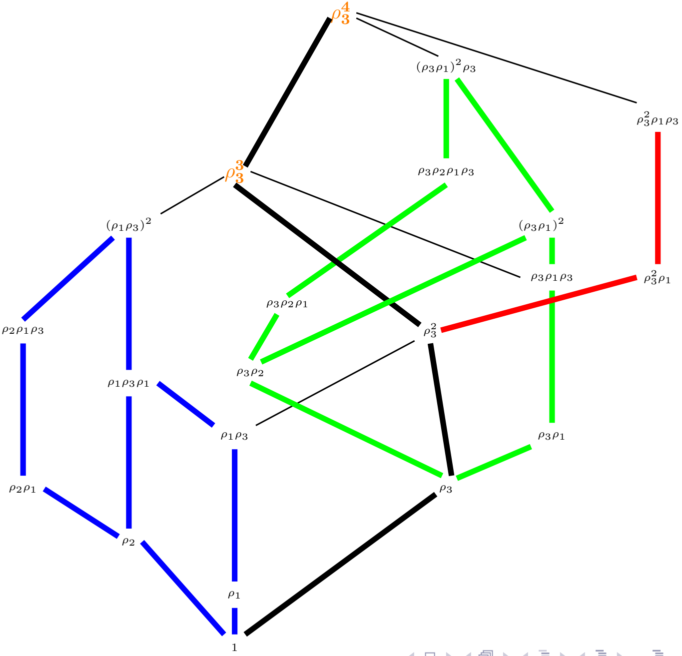
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ρ_3^2
$\rho_1\rho_3\rho_1$
$\rho_3\rho_2$
$\rho_1\rho_3$
$\rho_3\rho_1$
$\rho_2\rho_1$
ρ_3
ρ_2
ρ_1
1

Number of divisors of Δ

- Let $\mathrm{Div}(\Delta_n) := \coprod_{0 \leq i \leq n+1} D_n^i$, where $D_n^i = \{x \in \mathrm{Div}(\Delta_n) \mid \rho_n^i \leq x, \rho_n^{i+1} \not\leq x\}$. Note that $D_n^{n+1} = \rho_n^{n+1}$.

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Proposition

Let $n \geq 1$. Then we have the following isomorphisms of posets (where subposets of $\mathrm{Div}(\Delta_n)$ are ordered by the restriction of left-divisibility on $M(n)$)

- *Every D_n^i is an interval in $\mathrm{Div}(\Delta_n)$,*
- $\mathrm{Div}(\Delta_{n-1}) \cong D_n^0$, $\mathrm{Div}(\Delta_0) \cong D_n^{n+1} \cong \{\bullet\}$.
- *For all $1 \leq i \leq n$, $D_n^i \cong \mathrm{Div}(\Delta_{n-i})$.*

Number of divisors of Δ, II

Corollary

Let $n \geq 2$*, and let* $A_n := |\mathrm{Div}(\Delta_n)|$*. Then*

$$A_n = 2A_0 + 2A_{n-1} + \sum_{i=1}^{n-2} A_i. \qquad (1)$$

It follows that $A_n = F_{2n}$*, where* $F_0, F_1, F_2, \ldots$ *denotes the Fibonacci sequence* $1, 2, 3, 5, 8, \ldots$

Thank you for your attention!