Torus knot groups, Garside groups, complex reflection groups

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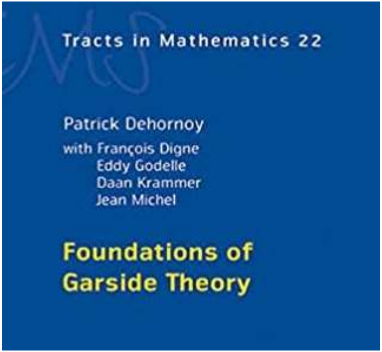
Tracts in Mathematics 22
Patrick Dehornoy
with François Digne
Eddy Godelle
Daan Krammer
Jean Michel
Foundations of
Garside Theory

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Let Σ_n be the submonoid of the n-strand braid group $\mathcal{B}_n$ generated by $\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{n-1}$.

Plan of the talk

Part I: **Dehornoy-Digne-Godelle-Krammer-Michel's monoid and some torus knot groups**

Part II: **A new Garside structure on torus knot groups and some braid groups of complex reflection groups**

Part III: **"Reflection" quotients of torus knot groups**

Part I

Dehornoy-Digne-Godelle-Krammer-Michel's monoid and some torus knot groups

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"Toric"

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"Dual"

$$\left\langle \tau_1, \tau_2, \tau_3 \,\middle|\, \tau_1\tau_2 = \tau_2\tau_3 = \tau_3\tau_1 \right\rangle$$

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- **Exotic monoid** : in terms of the classical Artin generators we have $a = \sigma_1$, $b = \sigma_1\sigma_2$. The Garside element Δ is given by b^3. We have $\text{Div}(\Delta) = \{1, a, b, ab, b^2, ba, bab, b^3\}$.

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- What about the "exotic" structure ?

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- If $n = 2$, we obtain the presentation

$$\left\langle \rho_1, \rho_2 \;\middle|\; \rho_1\rho_2\rho_1 = \rho_2^2 \right\rangle = \left\langle a, b \;\middle|\; aba = b^2 \right\rangle$$

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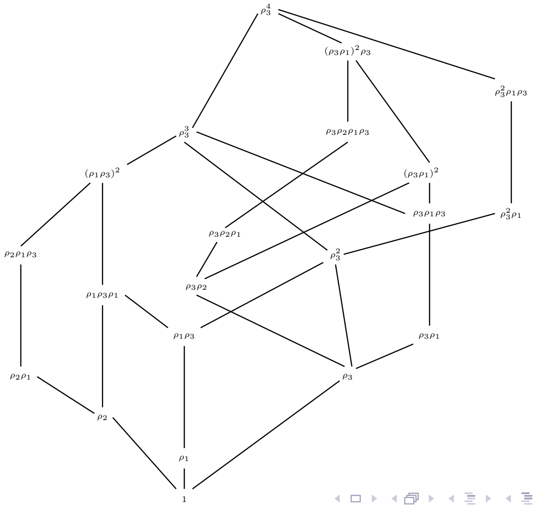

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Let $\Delta_n := \rho_n^{n+1}$ be the Garside element in $\mathcal{M}(n)$. We have

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(There is a description of the lattice of simples in terms of *Schroeder trees*).

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3. *We have $\mathcal{H}_{n+1} \cong \mathcal{B}_{n+1}$.*

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In particular, we can negatively answer the first part of DDGKM's question. It the above conjecture holds, then we can positively answer the second part of DDGKM's question.

Part II

A new Garside structure on torus knot groups and some braid groups of complex reflection groups

Torus knot groups and braid groups of complex reflection groups

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Theorem (Generalization to arbitrary torus knot groups)

Let $n, m \geq 2$ be two relatively prime integers. The Garside structure introduced above for $\mathcal{G}(n) \cong G(n, n+1)$ can be generalized to all torus knot groups $G(n, m)$.

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(we have $\mathcal{M}(n) = \mathcal{M}(n, n+1)$).

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$$\begin{aligned} G(2,3) \cong \mathcal{B}_3 \cong \mathcal{B}(G_4) \cong \mathcal{B}(G_8) \cong \mathcal{B}(G_{16}), \\ G(2,5) \cong \mathcal{B}(G_{20}), \\ G(2,m) \cong \mathcal{B}(I_2(m)),\ m \text{ odd}, \\ G(3,4) \cong \mathcal{B}(G_{12}),\ G(3,5) \cong \mathcal{B}(G_{22}). \end{aligned}$$

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- Moreover, a Garside structure analogous to the one introduced above for torus knot groups can be constructed for a few additional braid groups of complex reflection groups of rank two which are not isomorphic to torus knot groups.

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- The following monoid yields a Garside monoid for G_{13}:

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- It was shown by Bannai that $\mathcal{B}(G_{13}) \cong \mathcal{B}(I_2(6))$.

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Part III

"Reflection" quotients of torus knot groups

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- Every finite reflection group occurring above is a rank-two complex reflection group.

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- This result is reminiscent of the following theorem : let W be a Coxeter group. Then W is a real reflection group if and only if W is finite.

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- In this way, we associate a Garside group to every J-group in the above defined family, in such a way that whenever the J-group is finite, one recovers the braid group of the CRG.

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- In fact the parameter k_i is the number of conjugacy classes of reflections in $W(k_i,n_i,m_i)$. Hence at least $W(k_1,n_1,m_1) \cong_{\text{ref}} W(k_2,n_2,m_2) \Rightarrow k_1 = k_2$.

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$$W_{k,n,m} = \left\langle r_1, r_2, r_3 \;\middle|\; \begin{array}{c} r_1^2 = r_2^2 = r_3^2 = 1, \\ (r_1 r_2)^k = (r_2 r_3)^n = (r_3 r_1)^m = 1 \end{array} \right\rangle$$

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- There is a group morphism $J := J\begin{pmatrix} k & n & m \\ & & \end{pmatrix} \longrightarrow W_{k,n,m}$, $s \mapsto r_1 r_2$, $t \mapsto r_2 r_3$, $u \mapsto r_3 r_1$.

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2	3	5	G_{22}	$W(H_3)^+ \cong \mathfrak{A}_5$
3	2	3	G_4	$W(A_3)^+ \cong \mathfrak{A}_3$
4	2	3	G_8	$W(B_3)^+ \cong \mathfrak{S}_4$
5	2	3	G_{16}	$W(H_3)^+ \cong \mathfrak{A}_5$
3	2	5	G_{20}	$W(H_3)^+ \cong \mathfrak{A}_5$
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- With the observations above, to show that $W(k,n_1,m_1) \cong_{\text{ref}} W(k,n_2,m_2)$ ($n_i < m_i$ and coprime) implies $(n_1,m_1)=(n_2,m_2)$, it suffices to show that $W^+_{k,n_1,m_1} \cong W^+_{k,n_1,m_1}$ implies that $(n_1,m_1)=(n_2,m_2)$.

Thank you for your attention !