Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

III. Positivity in Temperley-Lieb algebras and dual Garside structures on Artin-Tits groups

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- The topological definition of Mikado braids is, as we will see today, useful and even necessary in some cases to show results involving Mikado braids.
- **Question:** Is there a topological characterization of Mikado braids in the above mentioned cases ?

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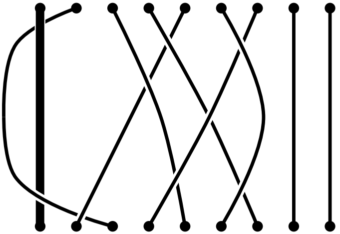
First model: Artin braids on $n+1$ strands with an unbraided first strand.



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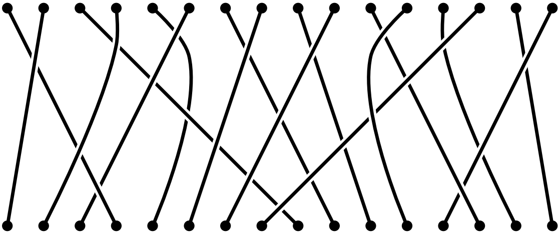
Second model: symmetric braids on $2n$ strands



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Let $\beta \in B(W^\Gamma)$. The following are equivalent

1. *The braid β is a Mikado braid.*
2. *There is an Artin braid in $B(W)$ representing β, such that one can inductively remove pairs of symmetric strands, one of the two strands being above all the other strands (so that the symmetric one is under all the other strands).*

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Example (A Mikado braid in type B_8)

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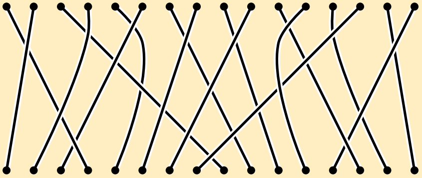

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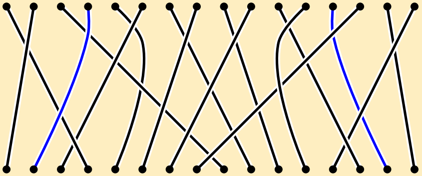

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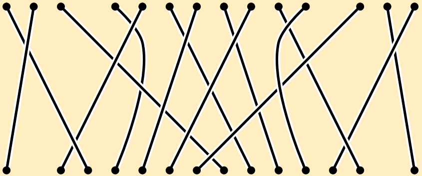

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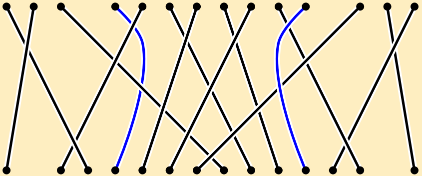

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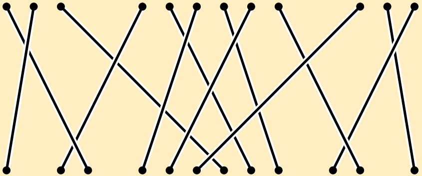

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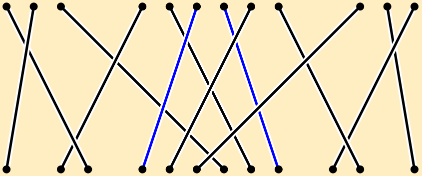

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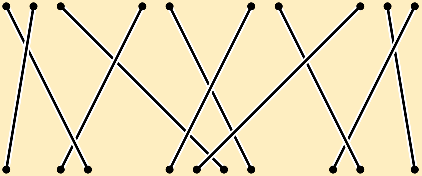

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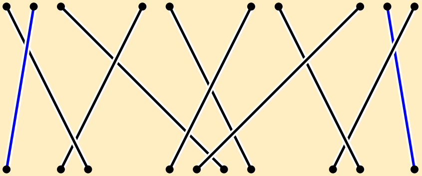

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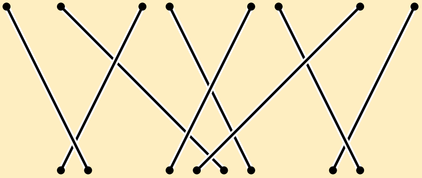

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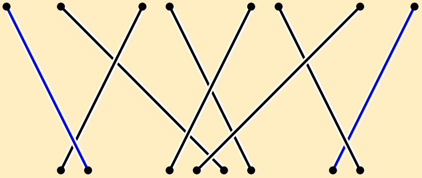

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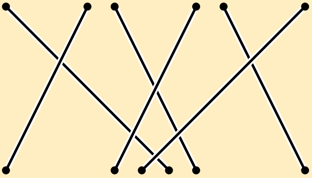

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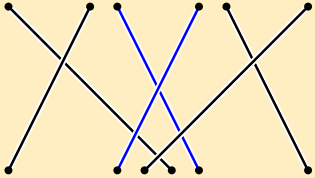

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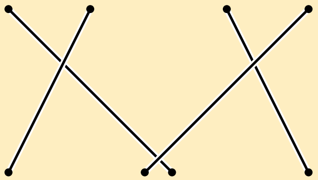

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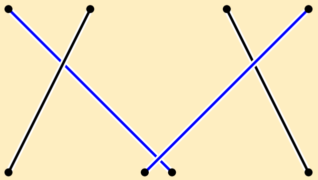

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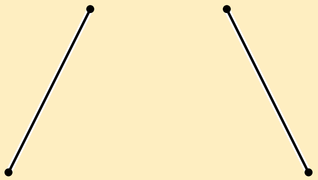

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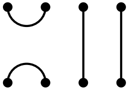
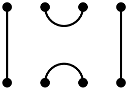
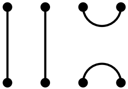
- Set $\delta := v + v^{-1}$.

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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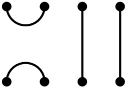
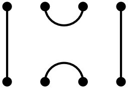
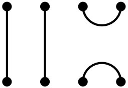
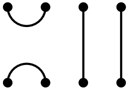
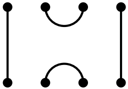
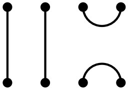
$b_1 =$  , $b_2 =$  , $b_3 =$ 

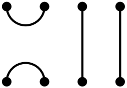
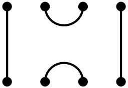
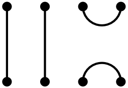
- Multiplication = concatenation

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

- Multiplication = concatenation
- Multiplication by δ = add a circle in the diagram

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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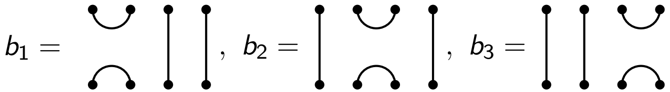
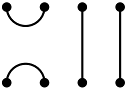
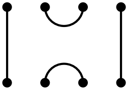
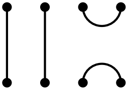


- Multiplication = concatenation
- Multiplication by δ = add a circle in the diagram

$$b_1 b_2 b_1 = b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

- Multiplication $=$ concatenation
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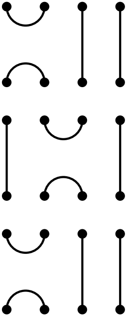
b_1

b_2

b_1



$$b_1 b_2 b_1 = b_1$$

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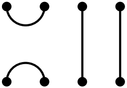
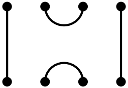
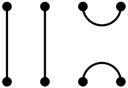
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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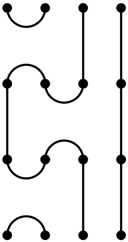
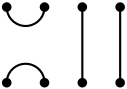
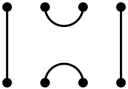
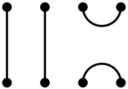


$$b_1 b_2 b_1 = b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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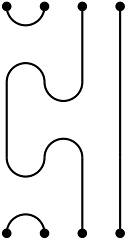
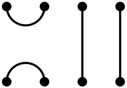
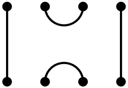
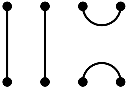


$$b_1 b_2 b_1 = b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

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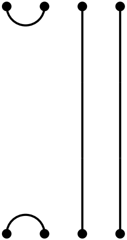
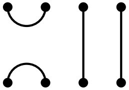
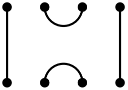
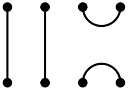


$$b_1 b_2 b_1 = b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

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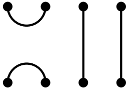

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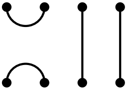
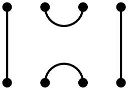
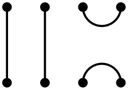
Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

- Multiplication = concatenation
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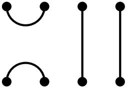
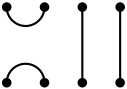
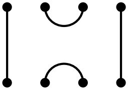
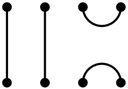
b_1 

$$b_1 b_2 b_1 = b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

- Multiplication = concatenation
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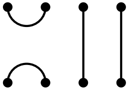
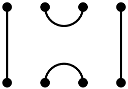
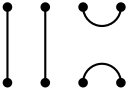
$$b_3 b_1 = b_1 b_3$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

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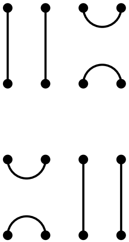
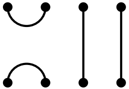
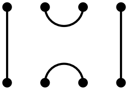
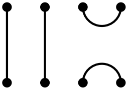
b_3

b_1



$$b_3 b_1 = b_1 b_3$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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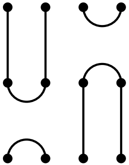
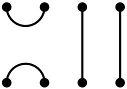
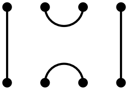
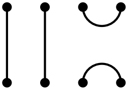


$$b_3 b_1 = b_1 b_3$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

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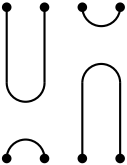
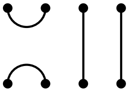
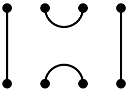
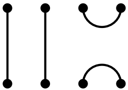


$$b_3 b_1 = b_1 b_3$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$ , $b_2 =$ , $b_3 =$ 

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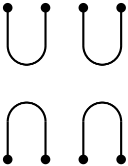

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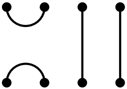
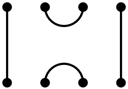
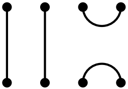
Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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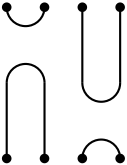

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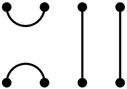
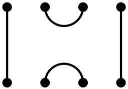
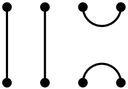
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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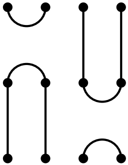

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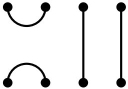
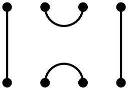
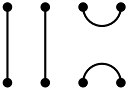
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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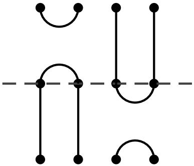

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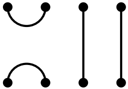
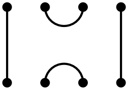
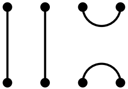
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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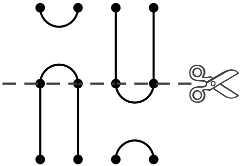

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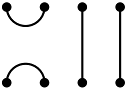
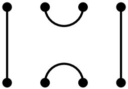
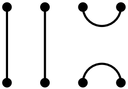
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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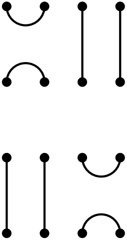

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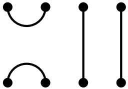
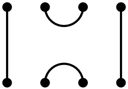
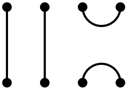
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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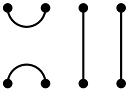
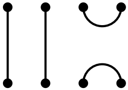
b_1 

b_3 

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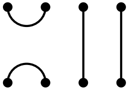
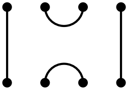
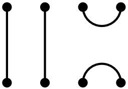
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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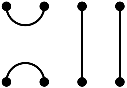
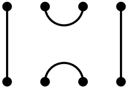
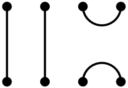
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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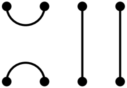
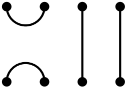
b_1 

b_1 

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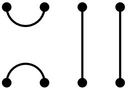
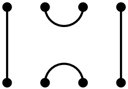
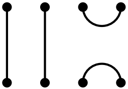
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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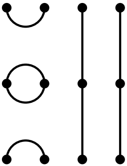

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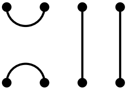
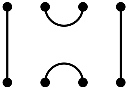
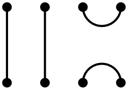
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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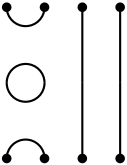

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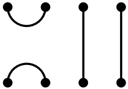
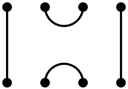
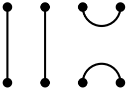
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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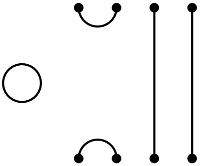

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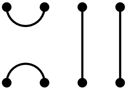
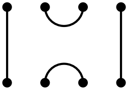
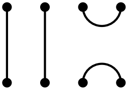
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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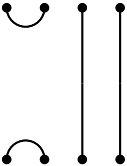
δ 

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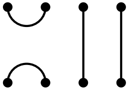
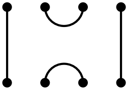
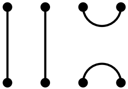
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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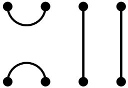
δ 

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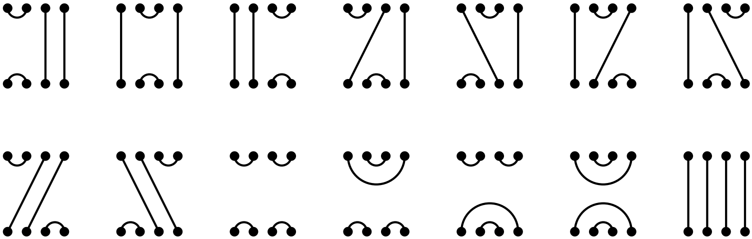

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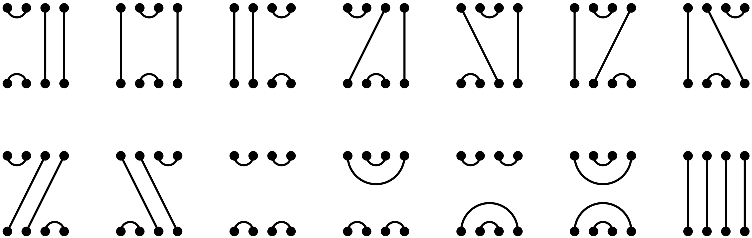

These diagrams, which form a basis of the obtained diagram algebra, form a basis of the algebra.

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- Let x be fully commutative, let $s_{i_1} s_{i_2} \cdots s_{i_i}$ be a reduced expression. As a consequence of the TL defining relations, the element

$$b_x := b_{i_1} b_{i_2} \cdots b_{i_k}$$

is well-defined.

Monomial or Kazhdan-Lusztig basis

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- In a monoid M with Noetherian divisibility, the (left-)divisibility $\leq$ is a partial order on M. Hence we can speak about least common (left-)multiples and greatest common (left-)divisors in such a monoid (if they exist !).

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- Note that in a monoid M with Noetherian divisibility, every nontrivial element has infinite order and there are no nontrivial invertible elements.

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- Under the above assumptions, one can define a group $G(M)$ of left-fractions of M, that is, whose elements are $f^{-1}g$ for $f, g \in M$, in which M embeds.

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Theorem (Garside, 1967)

The Artin-Tits group $B(W)$ is a Garside group, with corresponding Garside monoid $(B(W)^+, \Delta)$ and we have $\{\text{Div. of } \Delta\} = \{\mathbf{x}, x \in W\}$.

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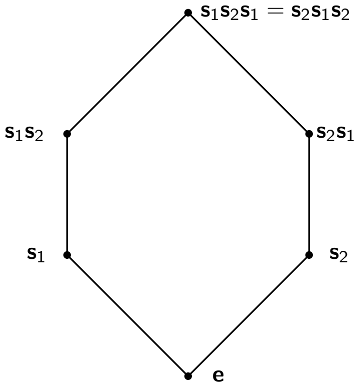

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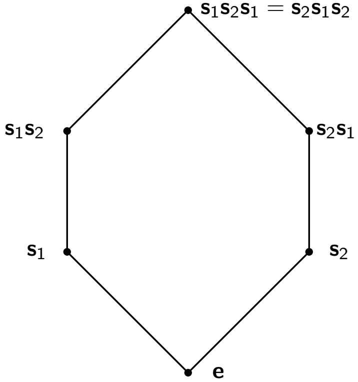

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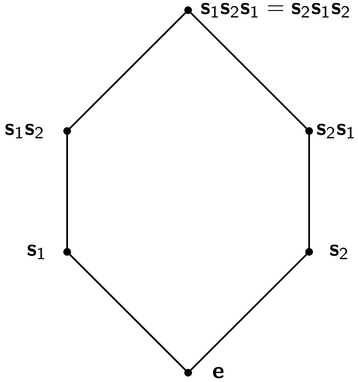

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- The set $\mathbf{S}$ is contained in the set of left divisors of Δ; these divisors therefore generate M.

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- The idea of dual braid monoids is to replace the generating set S of W by the set T of all the reflections, and build a monoid as above.

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Definition (Coxeter element)

A *Coxeter element* is a product of all the elements of S, in some order. Is is an n-cycle.

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Proposition (Biane, 1997)

The poset $(P_c, \leq_T)$ is a lattice, isomorphic to the noncrossing partition lattice.

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Definition

A *noncrossing partition* is a partition π of $\{1, 2, \ldots, n\}$ such that the following never happens: B_1, B_2 are two distinct blocks of π with $i, j \in B_1$, $k, \ell \in B_2$ and $i < k < j < \ell$.

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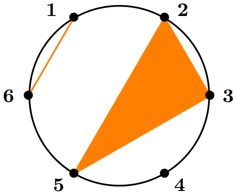

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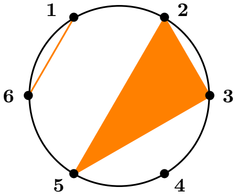

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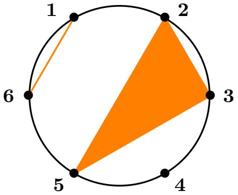

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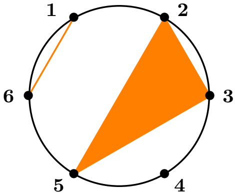

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- Let W be a finite Coxeter group with set of reflections T and c be a Coxeter element. Let $P_c := \{x \in W \mid x \leq_T c\}$. It can be shown that $T \subseteq P_c$ for all c.

Definition (dual braid monoids)

The *dual braid monoid* B_c^* is defined by

$$B_c^* = \langle x_c, x \in P_c \mid x_c(x^{-1}y)_c = y_c \text{ if } x \leq_T y \rangle.$$

Theorem (Bessis, 2004)

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- **Question**: is there a nice formula for $\iota(x_c)$ in the generators $\mathbf{S}$?

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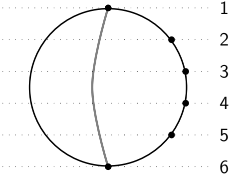

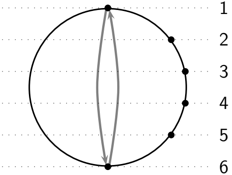

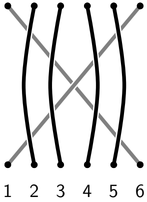

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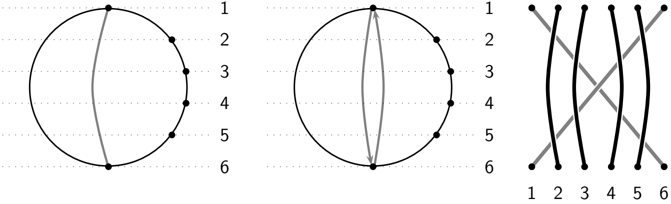

- For every $x \in P_c$, the Artin braid x_c is obtained first by taking a T-reduced expression $x = t_1 t_2 \cdots t_k$, $t_i \in T$ and then we have $x_c = (t_1)_c (t_2)_c \cdots (t_k)_c$.

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- **Open problem**: find a combinatorial proof of the above positivity theorem.

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Proof.

We have seen that x_c is Mikado. By Dyer's positivity, the result follows. □

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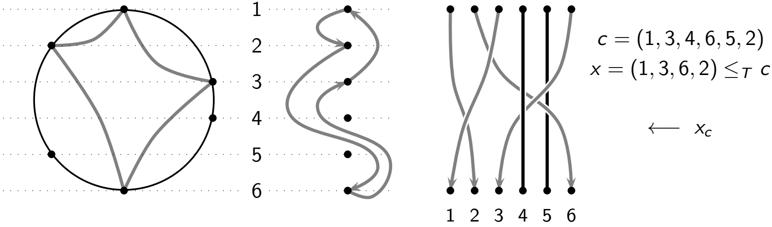

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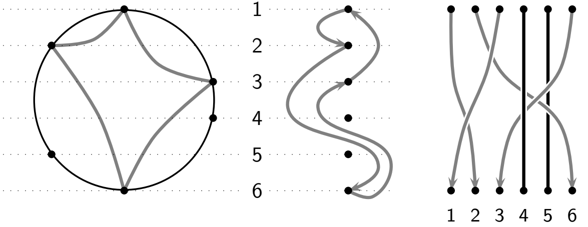

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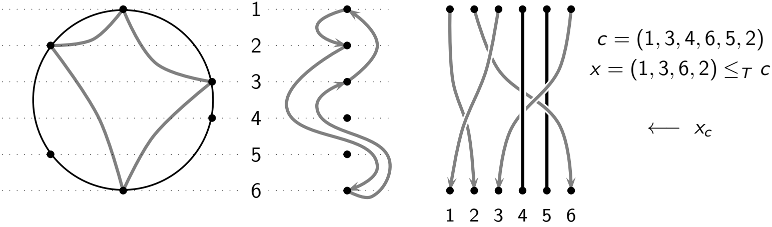

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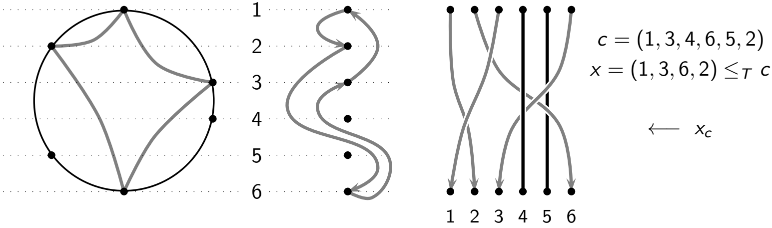

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- In type D_n, there are more simple elements than the dimension of the Temperley-Lieb algebra.

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- Do the images of the simple dual braids in the Temperley-Lieb algebras of types B_n or D_n have nice positivity properties ?
- There are (quasi-)Garside structures on $B(W)$ in case W has type $\widetilde{A}_n$ or $\widetilde{C}_n$ (warning: only for certain choices of Coxeter element).

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References

- B. Baumeister and T. Gobet, Simple dual braids, noncrossing partitions and Mikado braids of type D_n, *BLMS* (2017).
- F. Digne and T. Gobet, Dual braid monoids, Mikado braids and positivity in Hecke algebras, *Math. Z.* (2017).
- T. Gobet, Noncrossing partitions, fully commutative elements and bases of the Temperley-Lieb algebra, J. Knot Theory Ramif. **25** (2016), no. 6, 27pp.
- T. Gobet, Coxeter-Catalan combinatorics and Temperley-Lieb algebras, preprint (2016), Arxiv 1608.04530.
- T. Gobet, Simple dual braids and c-sortable elements, in progress.
- T. Licata and H. Queffelec, Braid groups of type ADE, Garside structures, and the categorified root lattice, preprint (2017), Arxiv 1703.06011.
- M.G. Zinno, A Temperley-Lieb basis coming from the braid group, J. Knot Theory Ramif. **11** (2002), 575-599.

THANK YOU !