Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

II. Generalized Kazhdan-Lusztig polynomials and Dyer's positivity conjectures

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If you are not familiar with the machinery of Coxeter / Artin groups, just keep in mind the case $W = \mathfrak{S}_n$, $B(W) = \mathcal{B}_n$ and the topological description of Mikado braids.

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- As a consequence of the definition, the H_s generate $\mathcal{H}(W)$, are invertible and satisfy the braid relations of W. In particular, there is a group homomorphism $\varphi : B(W) \longrightarrow \mathcal{H}(W)^\times$, $\mathbf{s} \mapsto H_s$, $s \in S$.

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3. *Every reduced expression of y has a reduced expression of x as a subword.*

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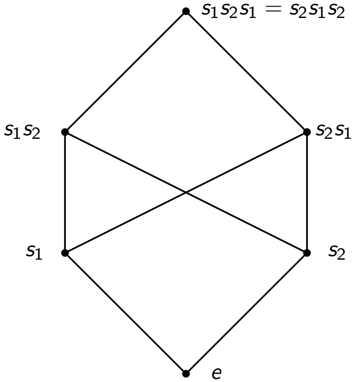

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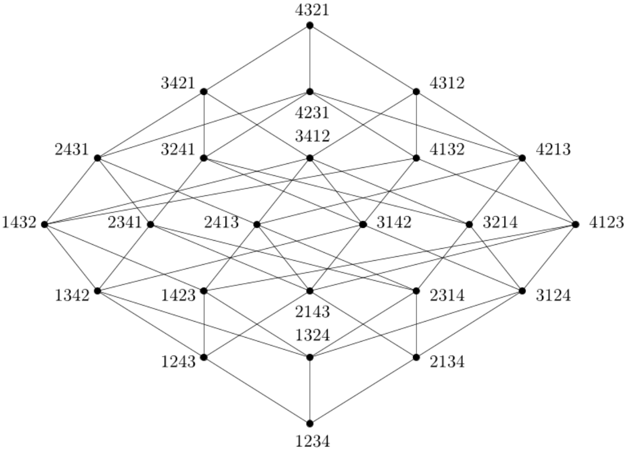

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- In fact the two sums above can be taken over all elements lower than w for the Bruhat order.
- The two sets $\{C_w\}_{w\in W}$ and $\{C'_w\}_{w\in W}$ yield two bases of $\mathcal{H}(W)$ called *canonical bases*.

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- Kazhdan and Lusztig also conjecture that $h_{y,w}$ lies in $\mathbb{Z}_{\geq 0}[v]$. This became known as *Kazhdan-Lusztig positivity conjecture*. Here there is no restriction on W.

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- Geometric proofs for (finite) Weyl groups were given much before: by KL 1980 for KL positivity, by Dyer-Lehrer 1990 for Dyer's conjecture.

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Here $C_y = ByB/B$ is a Schubert cell in the flag variety $X = G/B$ and $X_w = \overline{C_w}$. Hidden behind this formula is a categorification of the Hecke algebra by perverse sheaves on X.

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- There is (a priori) no recourse to geometry of Schubert varieties or category $\mathcal{O}$ for general Coxeter groups W. This raises a natural question: *Is there a framework in which to prove KL positivity in general ? Is there some "representation theory" in which KL polynomials can be interpreted as (graded) multiplicities ?*

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- For $s \in S$, set $B_s := R \otimes_{R^s} R(1)$, where $R^s = \{f \in R \mid s(f) = f\}$. It is an (indecomposable) graded R-bimodule. The category of graded R-bimodules is Krull-Schmidt, that is, every graded R-bimodule decomposes in an essentially unique way in a direct sum of indecomposables.

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- Given an additive category $\mathcal{C}$, we define its *split Grothendieck group* $\langle \mathcal{C} \rangle$ as the abelian group generated by symbols $\langle M \rangle$ for every object $M \in \mathcal{C}$ (modulo isomorphisms) with relations $\langle M \rangle = \langle M' \rangle + \langle M'' \rangle$ whenever $M \cong M' \oplus M''$.

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If moreover the bimodules are graded, then $\langle \mathcal{C} \rangle$ is even an $\mathcal{A}$-algebra, where the operation of v is defined by

$$v \cdot \langle M \rangle := \langle M(1) \rangle.$$

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Conjecture (Soergel 2007; proven by Elias and Williamson 2014)

$\mathcal{E}(C'_w) = \langle B_w \rangle$ *for all* $w \in W$.

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&\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\
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&\cong (R \otimes_{R^s} R) \oplus (R \otimes_{R^s} R)(2) \\
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- We have checked the isomorphism $\mathcal{H}(W) \cong \langle \mathcal{B} \rangle$ in that case.

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Proposition ("Standard filtrations", Soergel, 2007)

Let $w_0 = e, w_1, w_2, \dots$ *be an enumeration of* W *refining* $\leq$. *For* $x \in W$,

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$$0 = B^k \subsetneq B^{k-1} \subseteq B^{k-2} \cdots \subseteq B^1 \subseteq B^0 = B$$

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Let $A \subseteq T$. Define a preorder $\leq_A$ on W as the transitive closure of $x <_A xt$ whenever $x \in W$, $t \in T$, $t \notin N(x^{-1}) + A$ (where $+$ means symmetric difference).

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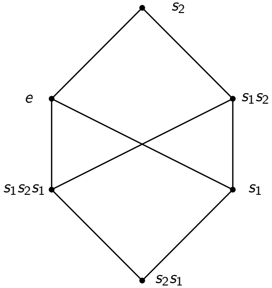

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- **Open question:** is there an interpretation of $h^A_{x,w}$ in the framework of (graded versions of the principal block of) category $\mathcal{O}$, in case W is a finite Weyl group ? (multiplicities of twisted Verma modules ?).

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- The H_s "should" be categorified by the complex

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in a suitable category (μ is the multiplication).

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- Every complex C in $K^b(\mathcal{B})$ admits a *minimal complex* $C^{\min}$, that is, with no contractible summand of the form $0 \to M \xrightarrow{\text{isom.}} M' \to 0$. This complex is unique up to isomorphism of complexes.

Theorem (G., 2016)

Let $x, y \in W$, $A := N(y)$ or $T \backslash N(y)$, $H_{x,A} = \sum_{w \in W} q_{x,w}^A C_w$.

1. *Let $w \in W$. The bimodule B_w appears as a direct summand in $C_{x_A}^{\min}$ either only in odd cohomological degrees or only in even degrees.*

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1. *Let $w \in W$. The bimodule B_w appears as a direct summand in $C^{\min}_{x_A}$ either only in odd cohomological degrees or only in even degrees.*
2. *The coefficient $q^A_{x,w}$ gives the multiplicity of B_w in all cohom. degrees of $C^{\min}_{x_A}$ together. $\Rightarrow q^A_{x,w} \in \mathbb{Z}[v^{\pm 1}]$.*

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