Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

I. Mikado braids

Thomas Gobet

Institut Elie Cartan de Lorraine, Nancy

Junior Hausdorff Trimester
"Symplectic Geometry and Representation Theory"
Bonn, October 2017

Classical braid group

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- Artin braid group on n strands: $\mathcal{B}_n :=$

$$\left\langle \mathbf{s}_1, \ldots, \mathbf{s}_{n-1} \;\middle|\; \begin{array}{cc} \mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_i = \mathbf{s}_{i+1}\mathbf{s}_i\mathbf{s}_{i+1} & i \leq n-2, \\ \mathbf{s}_i\mathbf{s}_j = \mathbf{s}_j\mathbf{s}_i & |i-j| > 1. \end{array} \right\rangle$$

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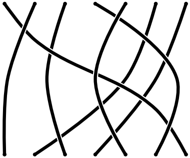
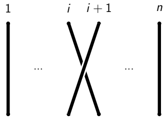


$\in \mathcal{B}_n,$



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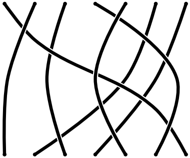

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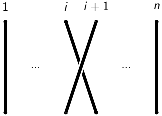
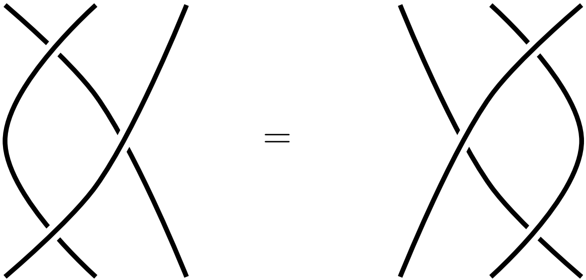


$=: \mathbf{s}_i.$

- Symmetric group on n letters: $\mathfrak{S}_n :=$

$$\left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{matrix} s_i^2 = 1 & \forall i \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} & i \leq n-2, \\ s_is_j = s_js_i & |i-j| > 1. \end{matrix} \right\rangle$$

Braid relation: some topological observations



$$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1 = \mathbf{s}_2\mathbf{s}_1\mathbf{s}_2$$

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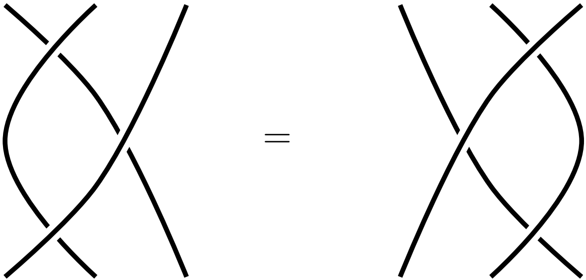

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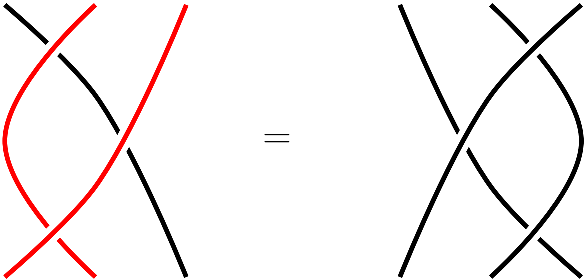

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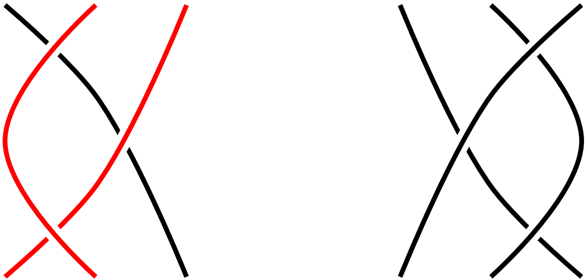

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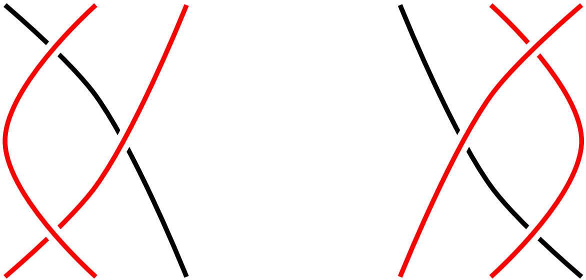

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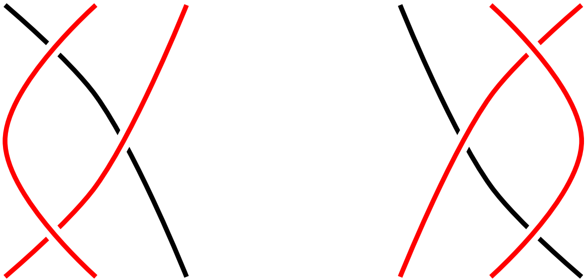

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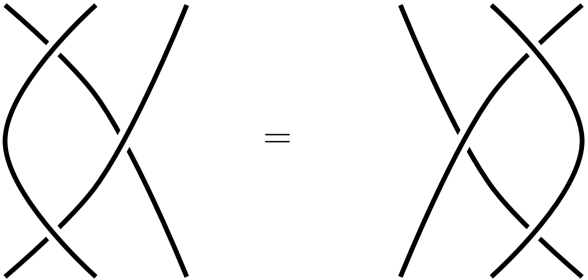

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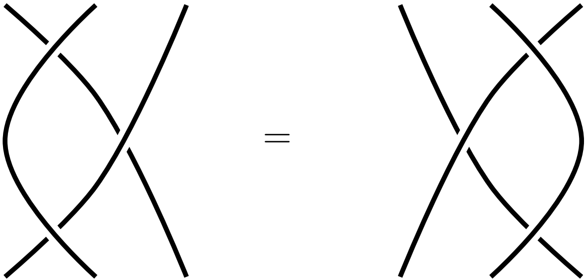

$$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1^{-1} = \mathbf{s}_2^{-1}\mathbf{s}_1\mathbf{s}_2$$

"Mixed" braid relation

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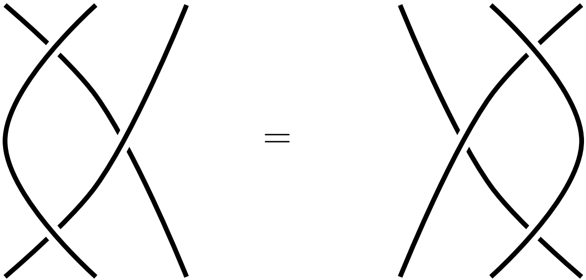

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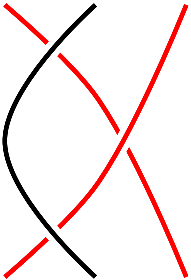

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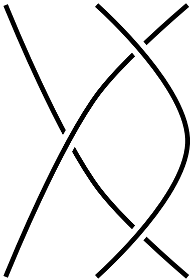

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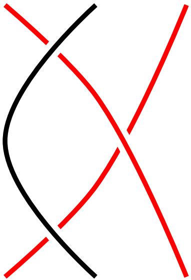

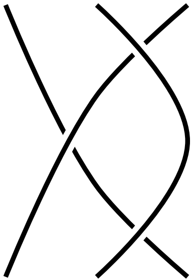

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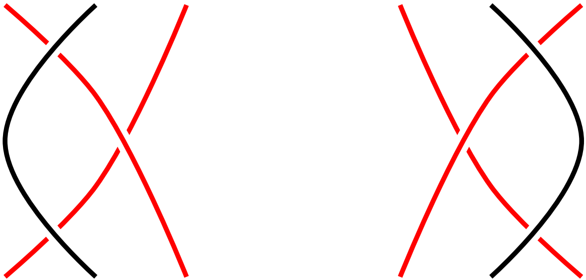

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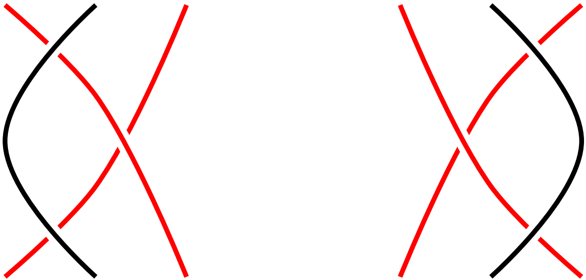

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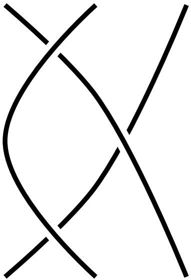

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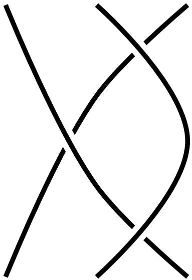

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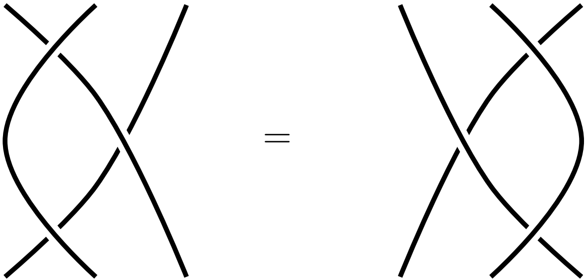

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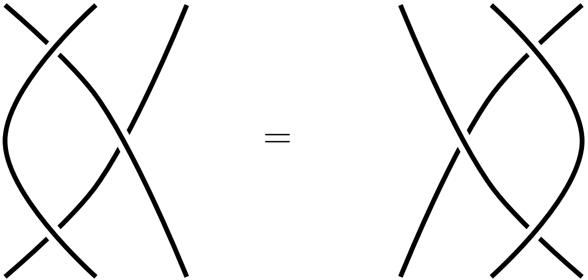

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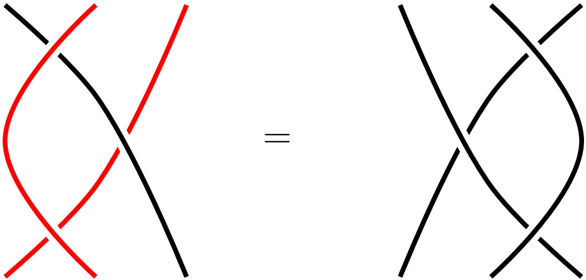

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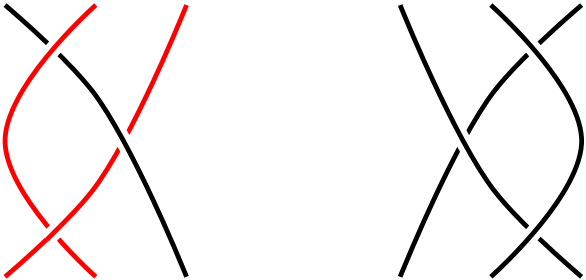

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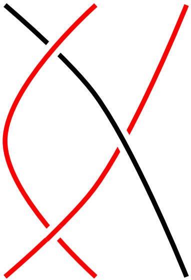

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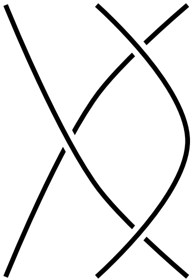

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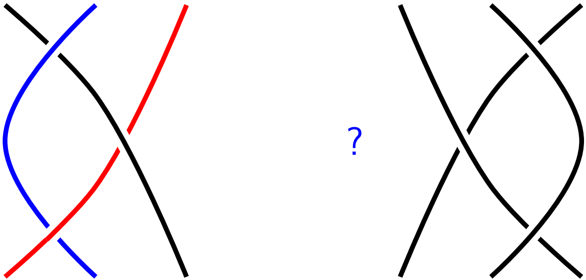

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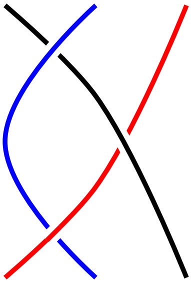

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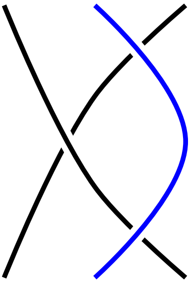

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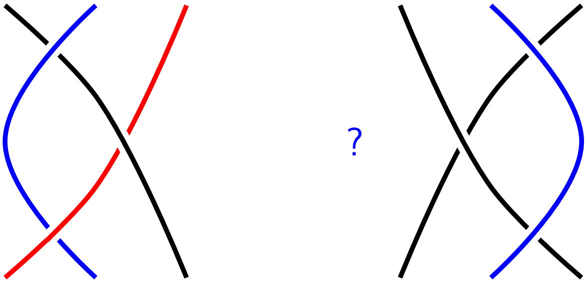

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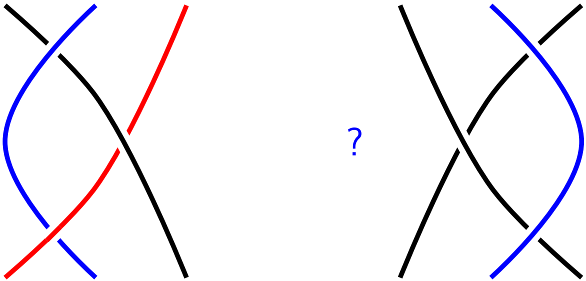

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$\Rightarrow$ Obstruction to a mixed braid relation.

Mixed braid relations

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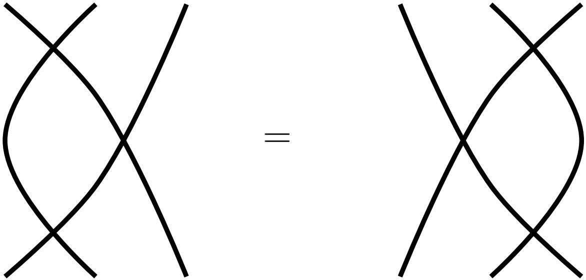

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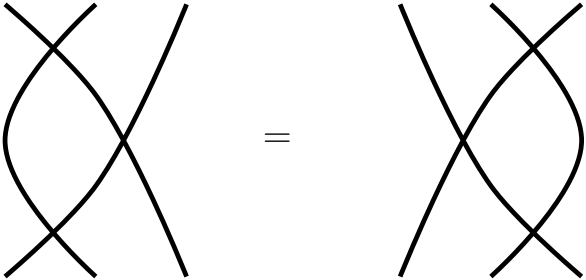
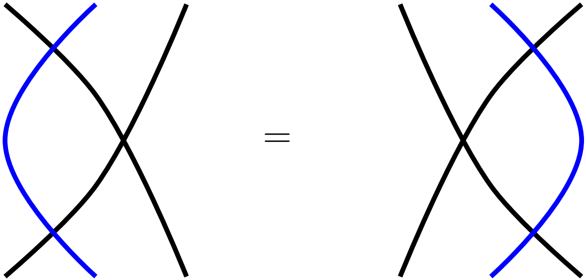


- What is the condition to put on the crossings / strands in the left braid above so that there exists a mixed braid relation?

Mixed braid relations



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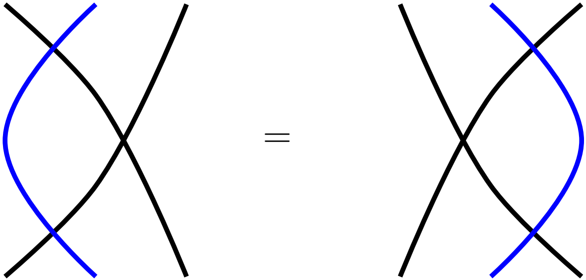

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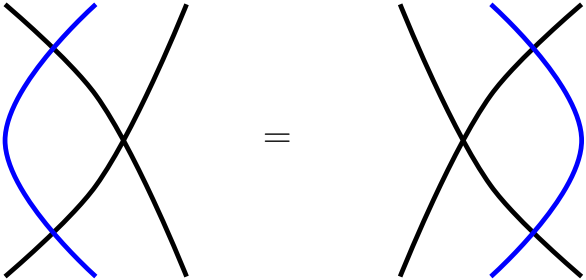

- What is the condition to put on the crossings / strands in the left braid above so that there exists a mixed braid relation?
- The blue strand should be either "above", "below" the crossing or "in between" the other two strands.
- In other words: you can remove all the strands of the braid, beginning by a strand which is above all the other strands, and going on in the same way.

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- If $\beta \in \mathcal{B}_{n+1}$ is Mikado, then it can be shown that removing *any* strand lying above all the others in *any* braid diagram for β will yield a Mikado braid in $\mathcal{B}_n$.

Mikado braid: example

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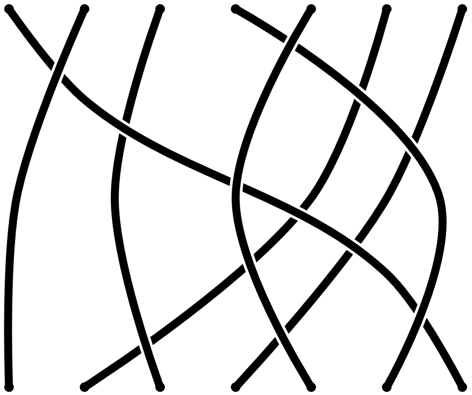

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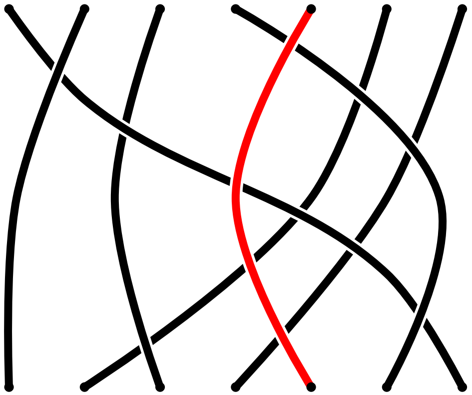

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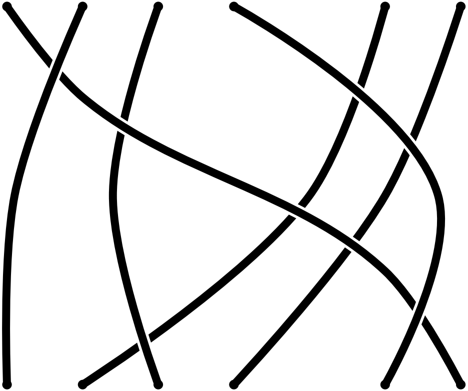

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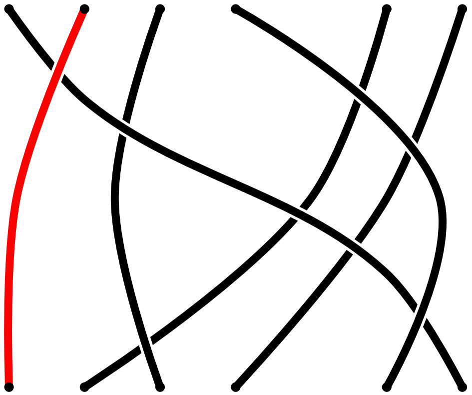

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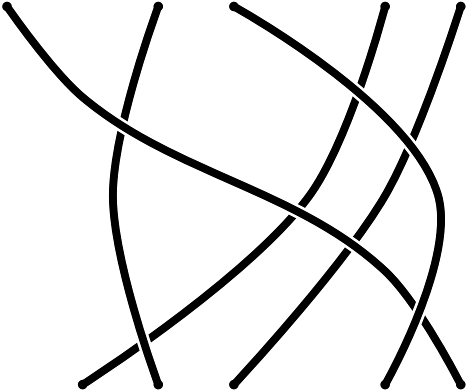

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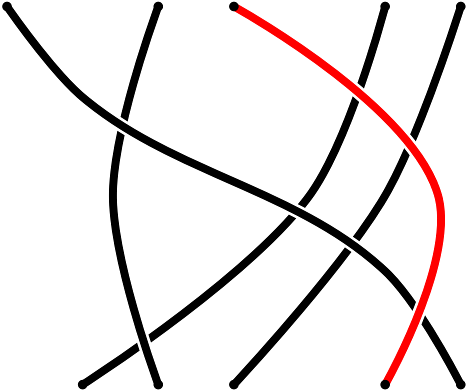

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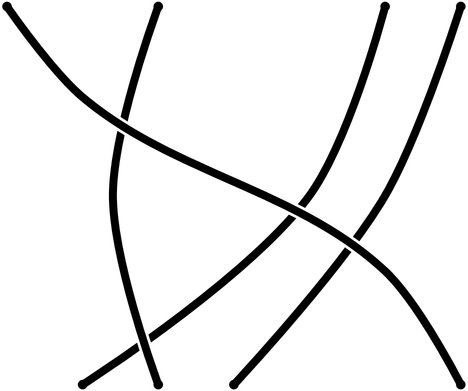

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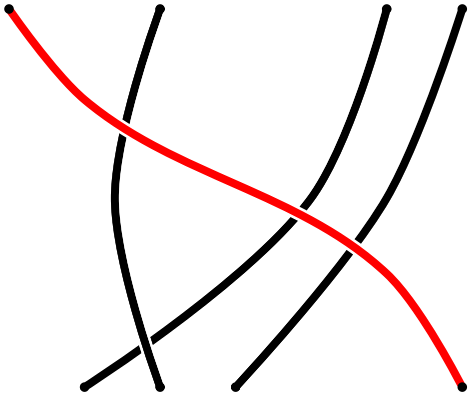

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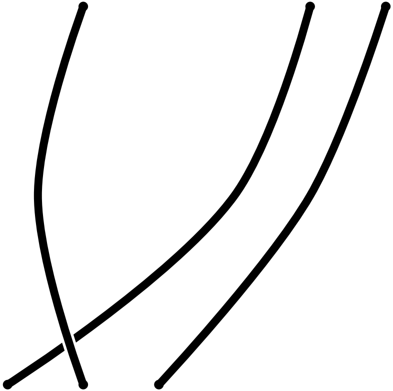

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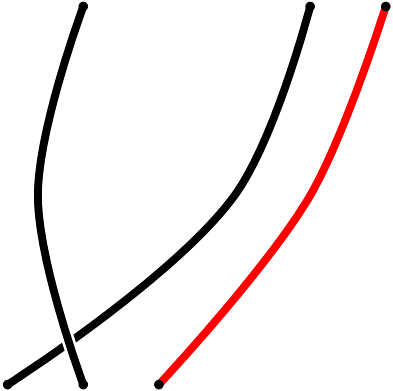

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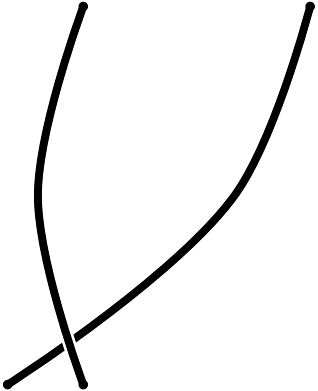

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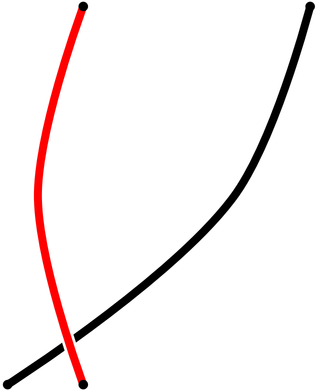

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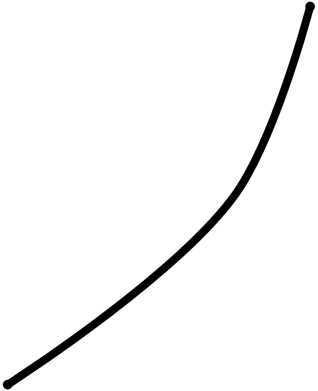

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Example: positive simple braids

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- Given a permutation $x \in \mathfrak{S}_n$, we say that a product $s_{i_1} s_{i_2} \cdots s_{i_k}$, where $i_1, i_2, \ldots, i_k \in \{1, \ldots, n-1\}$, is a *reduced expression* of x if $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ and k is minimal. The integer $\ell(x) := k$ is the *length* of x.

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- Every permutation $x \in \mathfrak{S}_n$ can be lifted to a *positive simple braid* (aka *canonical lift*) $\mathbf{x}$: choose a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ of x in the generating set S, then replace each s_{i_j} by $\mathbf{s}_{i_j}$.

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- Every permutation $x \in \mathfrak{S}_n$ can be lifted to a *positive simple braid* (aka *canonical lift*) $\mathbf{x}$: choose a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ of x in the generating set S, then replace each s_{i_j} by $\mathbf{s}_{i_j}$. The obtained element $\mathbf{x}$ is independent of the reduced expression thanks to the above Lemma.

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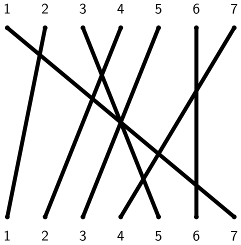

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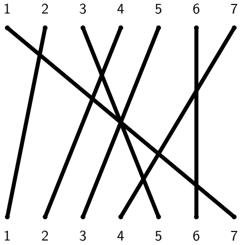

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- The number of crossings in D_x (counted with multiplicities !) is equal to $\ell(x)$ (in the example above we have $\ell(x) = 10$).

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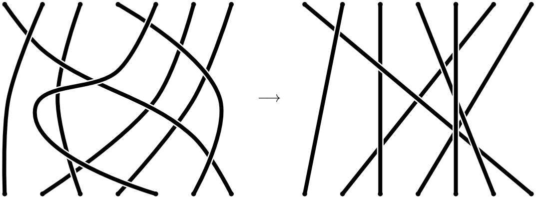

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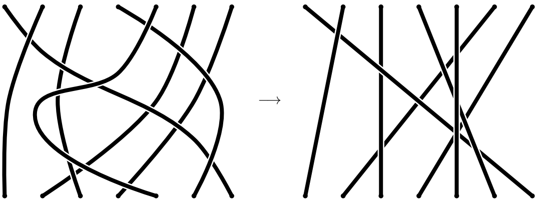

- This means that a Mikado braid can be obtained by "lifting a reduced expression of a permutation", i.e., replacing each generator s in the expression by $\mathbf{s}^{\pm 1}$.

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Definition (Square-free braids)

Let $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of a permutation. A braid of the form $\mathbf{s}_{i_1}^{\varepsilon_1} \mathbf{s}_{i_2}^{\varepsilon_2} \cdots \mathbf{s}_{i_k}^{\varepsilon_k}$ where $\varepsilon_j \in \{\pm 1\}$ for all j is a *square-free braid*.

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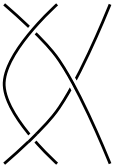

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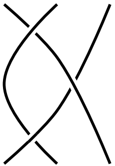

- **Question:** Can we characterize those square-free braids which are Mikado?

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- Starting from a longer reduced expression, for example $s_2s_3s_2s_1s_2s_3s_4s_3s_2s_1$, is there a way to determine which lifts are Mikado?

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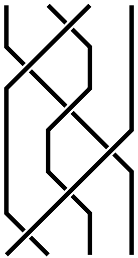

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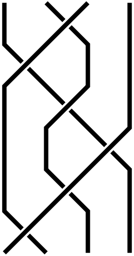

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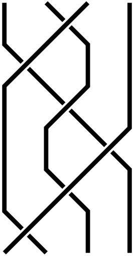

Mikado braids, Soergel bimodules, and positivity in Hecke and Temperley-Lieb algebras

I. Mikado braids

Thomas Gobet

The Artin braid group

Mikado braids

Characterizations of Mikado braids

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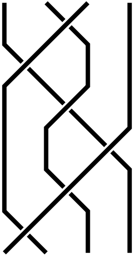

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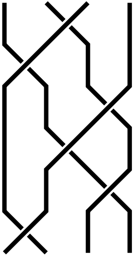

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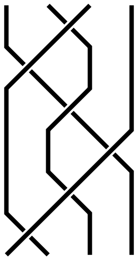

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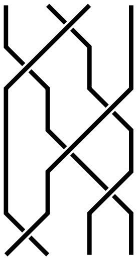

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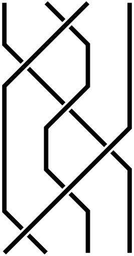

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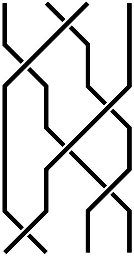

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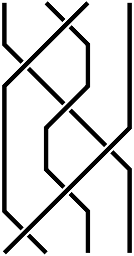

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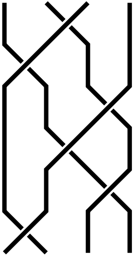

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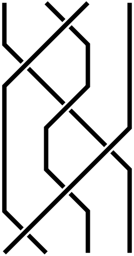

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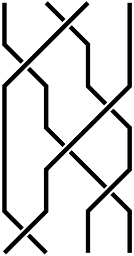

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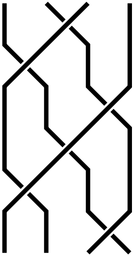

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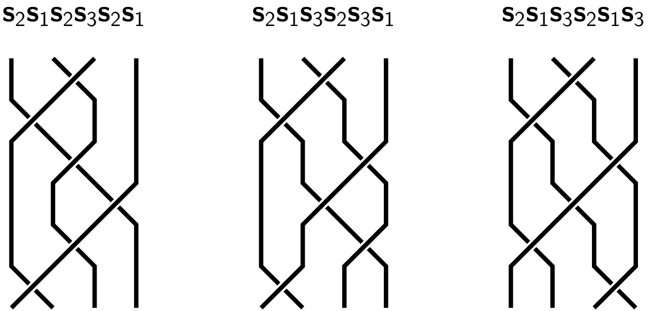

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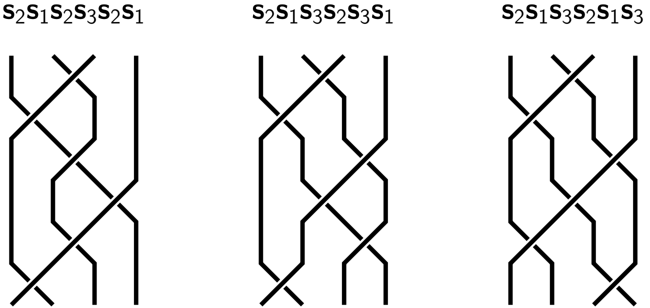

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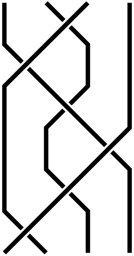

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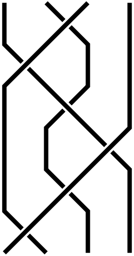

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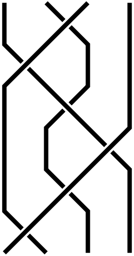

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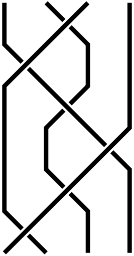

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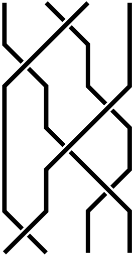

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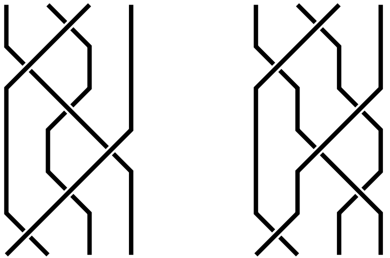

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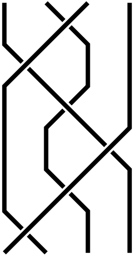

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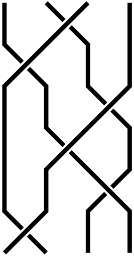

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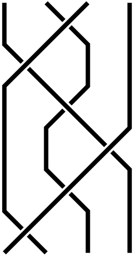

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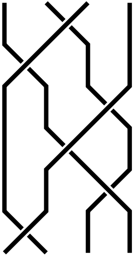

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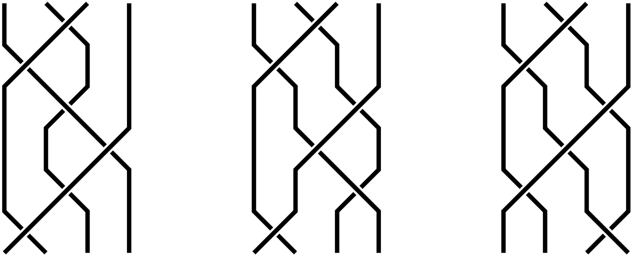

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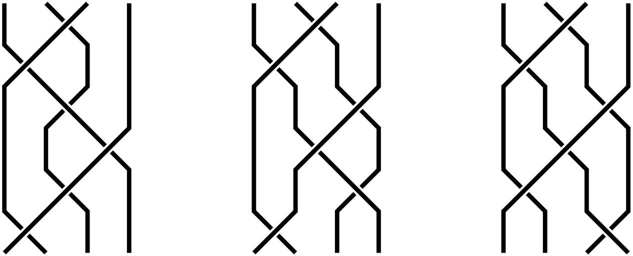

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Let $s_{i_1}s_{i_2}\cdots s_{i_k}$ *be a reduced expression of* $x \in \mathfrak{S}_n$. *Let* $\beta = \mathbf{s}_{i_1}^{\varepsilon_1}\mathbf{s}_{i_2}^{\varepsilon_2}\cdots\mathbf{s}_{i_k}^{\varepsilon_k}$, $\varepsilon_j = \pm 1$, *be a lift of* x. *Assume that* β *is Mikado. There is a bijection between the reduced expressions of* x *and those of* β;

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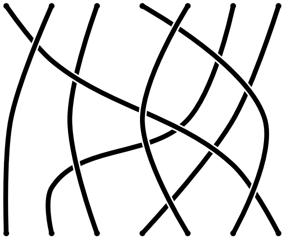

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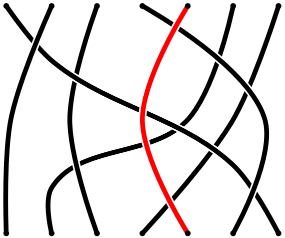

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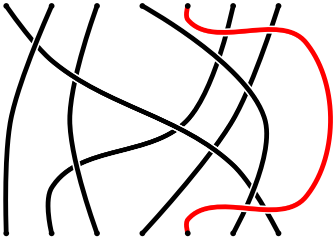

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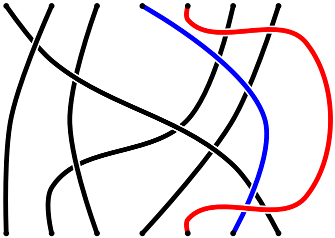

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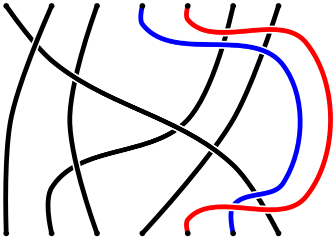

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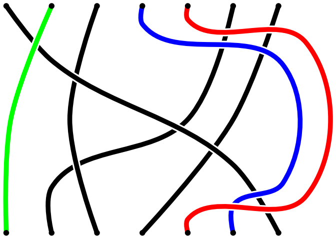

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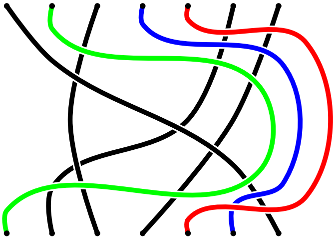

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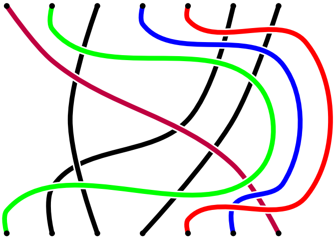

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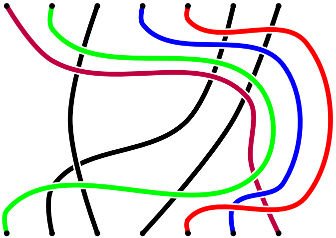

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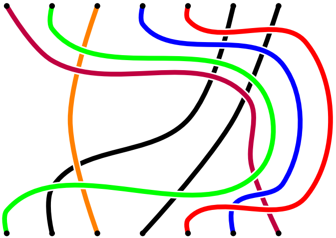

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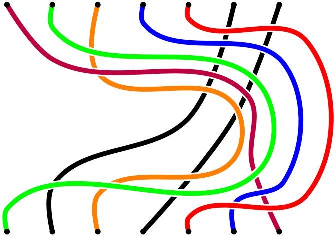

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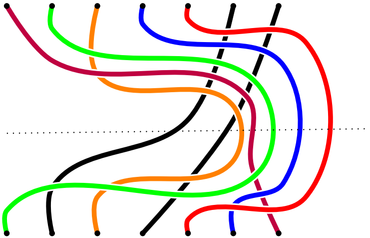

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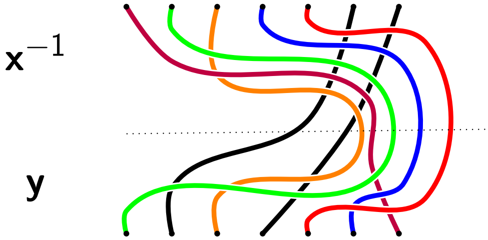

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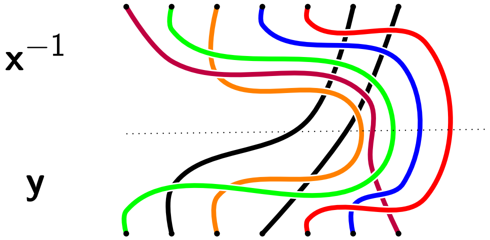

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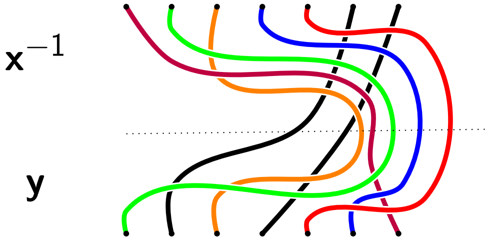

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Proposition

A braid $\beta \in \mathcal{B}_n$ is a Mikado braid if and only if there are $x, y \in \mathfrak{S}_n$ such that $\beta = \mathbf{x}^{-1}\mathbf{y}$, if and only if there are $u, v \in \mathfrak{S}_n$ such that $\beta = \mathbf{u}\mathbf{v}^{-1}$.

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$$x_{N(y)} := \mathbf{s}_{i_1}^{\varepsilon_1} \mathbf{s}_{i_2}^{\varepsilon_2} \cdots \mathbf{s}_{i_k}^{\varepsilon_k},$$

where $\varepsilon_j = -1$ if $s_{i_k} s_{i_{k-1}} \cdots s_{i_j} \cdots s_{i_{k-1}} s_{i_k} \in N(y)$ and 1 otherwise.

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Corollary

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5. *(Garside theoretic) We have $\Delta^{-1} \leq \beta \leq \Delta$, where Δ is the half twist (= the canonical lift of the unique longest permutation in $\mathfrak{S}_n$).*

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Symmetric group = Coxeter group

Braid group = Artin-Tits group

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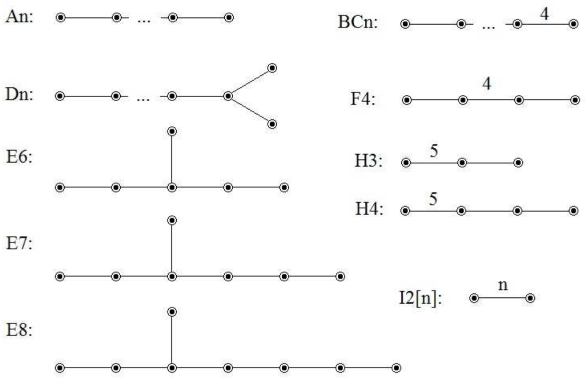
- For reasons which will become clear later, for the moment we do *not* want to restrict to finite Coxeter groups (equivalently spherical Artin-Tits groups).

Finite Coxeter groups



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- **Question:** Can we define a Mikado braid in a general Artin-Tits group?

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- In particular, if W is finite, then biclosed sets of roots are precisely inversion sets of elements.
- **Exercise:** Let W be the infinite dihedral group (i.e. $|S| = 2$, no braid relation). Show that the biclosed sets of roots are exactly inversion sets of elements, their complements, plus two infinite sets of roots which are complement to each other.

Mikado braids in general Artin-Tits groups

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Definition (Mikado braids in Artin-Tits groups)

Let W be a Coxeter group. We say that $\beta \in B(W)$ is a *Mikado braid* if there is $x \in W$ and a biclosed set $A \subseteq \Phi^+$ such that $\beta = x_A$.

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- The topological definition of Mikado braids is, as we will see further, useful and even necessary in some cases to show results involving Mikado braids.
- **Question:** Is there a topological characterization of Mikado braids in the above mentioned cases?

Topological models in the classical types: type B_n

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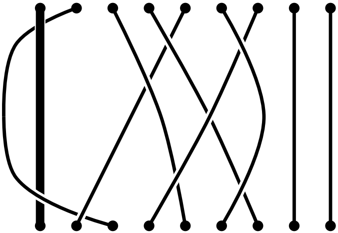
First model: Artin braids on $n+1$ strands with an unbraided first strand.



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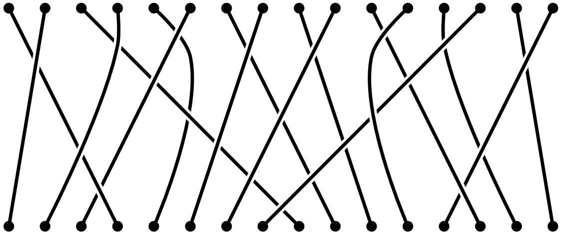
Second model: symmetric braids on $2n$ strands



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1. *The braid β is a Mikado braid.*
2. *There is an Artin braid in $B(W)$ representing β, such that one can inductively remove pairs of symmetric strands, one of the two strands being above all the other strands (so that the symmetric one is under all the other strands).*

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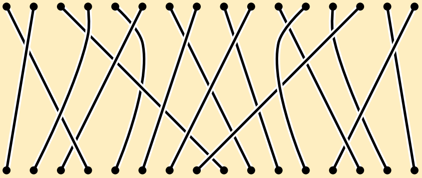
Example (A Mikado braid in type B_8)

Second model: symmetric braids on 2n strands



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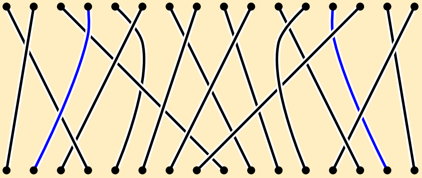

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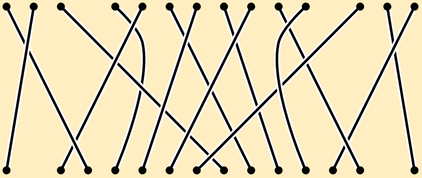

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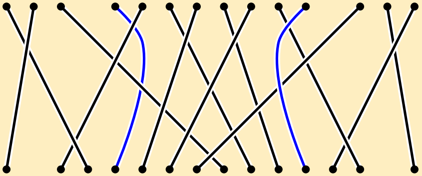

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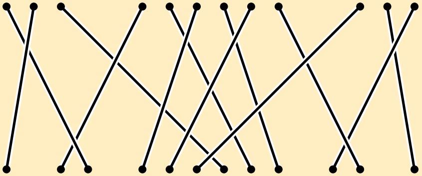

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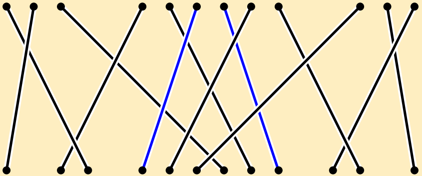

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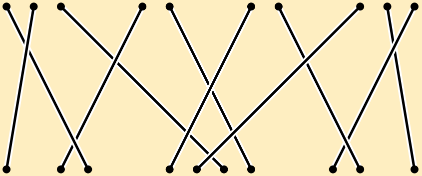

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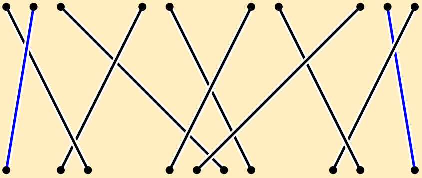

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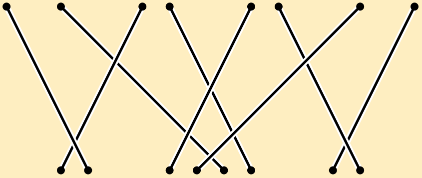

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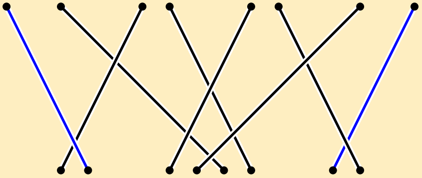

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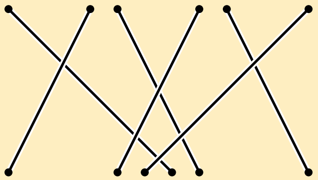

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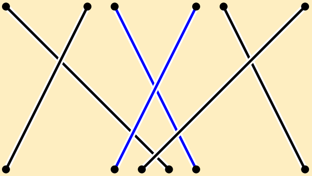

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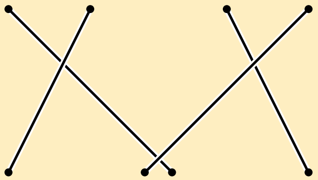

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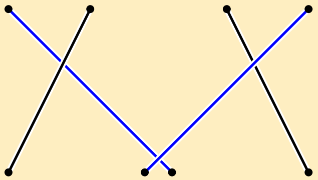

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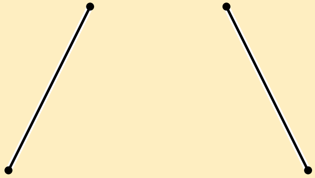

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Theorem (Baumeister-G. 2017)

Let $\beta \in B(W') \subseteq \overline{B}$. The following are equivalent.

1. *The braid β is Mikado.*
2. *There is a Mikado braid $\beta' \in B(W^\Gamma)$ such that $\beta = \pi(\beta')$, where $\pi : B(W^\Gamma) \twoheadrightarrow \overline{B}$ is the quotient map.*

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- **Open question:** is there a topological interpretation of Mikado braids in these cases?

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